Fundamentals of Fuzzy Systems

Classical logic and mathematics assume that we can assign one of the two values, *true* or *false*, to each logical proposition or statement. If a suitable formal model for a certain problem or task can be specified, conventional mathematics provides powerful tools which help us to solve the problem. When we describe such a formal model, we use a terminology which has much more stringent rules than natural language. This specification often requires more work and effort, but by using it we can avoid misinterpretations. Furthermore, based on such models we can prove or reject hypotheses or derive unknown correlations.

However, in our everyday life formal models do not concern the inter-human communication. Human beings are able to assimilate easily linguistic information without thinking in any type of formalization of the specific situation. For example, a person will have no problems to accelerate slowly while starting a car, if he is asked to do so. If we want to automate this action, it will not be clear at all, how to translate this advice into a well-defined control action. It is necessary to determine a concrete statement based on an unambiguous value, i.e. step on the gas at the velocity of half an inch per second. On the other hand, this kind of information will not be adequate or very helpful for a person.

Therefore, automated control is usually not based on a linguistic description of heuristic knowledge or knowledge from one’s own experience, but it is based on a formal model of the technical or physical system. This method is definitely a suitable approach, especially if there is a good model to be determined.

However, a completely different technique is to use knowledge formulated in natural language directly for the design of the control strategy. In this case, a main problem will be the translation of the verbal description into concrete values, i.e. assigning “step on the gas slowly” into “step on the gas at the velocity of a centimeter per second” as in the above mentioned example.

When describing an object or an action, usually use uncertain or vague concepts. In natural language we hardly ever find exactly defined concepts.
like supersonic speed for the velocity of a passing airplane. Supersonic speed characterizes an unambiguous set of velocities, because the speed of sound is a fixed entity and therefore it is unambiguously clear whether an airplane flies faster than sound or not. Frequently used vague concepts, like fast, very big, small and so on, make it impossible to decide unambiguously whether a given value satisfies such a vague concept or not. One of the reasons for this is that vague concepts are usually context dependent. Talking about airplanes fast has a different meaning than using this characteristic while referring to cars. But also if we agree that we are talking about cars it is not easy to distinguish clearly between fast and non-fast cars. The difficulty here is not to find a value telling us whether a car (or its top speed) is fast or not, but we had to presuppose that such a value does exist. It is more likely that we will be reluctant to fix such a value because there are velocities, we can classify as fast for a car and there are some we can classify as not fast, and in between there is a wide range of velocities which are considered as more or less fast.

1.1 Fuzzy Sets

The idea of fuzzy sets is to solve this problem by avoiding the sharp separation of conventional sets into two values - complete membership or complete non-membership. Instead, fuzzy sets can handle partial membership. So in fuzzy sets we have to determine to what degree or extend an element is a member of this fuzzy set. Therefore, we define:

**Definition 1.1** A fuzzy subset or simply a fuzzy set $\mu$ of a set $X$ (the universe of discourse) is a mapping $\mu : X \rightarrow [0, 1]$, which assigns to each element $x \in X$ a degree of membership $\mu(x)$ to the fuzzy set $\mu$. The set of all fuzzy sets of $X$ is denoted by $F(X)$.

A conventional set $M \subseteq X$ can be viewed as a special fuzzy set by identifying it with its characteristic function or indicator function.

$$I_M : X \rightarrow \{0, 1\}, \quad x \mapsto \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{else} \end{cases}$$

Seen in this way, fuzzy sets can be considered as generalized characteristic functions.

**Example 1.2** Figure 1.1 shows the characteristic function of the set of velocities which are higher than 170 km/h. This set does not represent an adequate model of all high velocities. The jump at the value of 170 causes that 169.9 km/h would not be a high velocity but 170.1 km/h would be. Therefore, a fuzzy set (figure 1.2) seems to be more adequate to model the concept high velocity.