1. Optimal Growth Without Discounting

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1.1 Introduction

As it is well known, the standard approach to infinite time horizon optimal growth problem is to discount future consumptions utilities by some factor and to maximize the resulting infinite series. Another approach pionnered by Ramsey [12] and reworked by Samuelson and Solow [13], Koopmans [9] and von Weizsäcker [14] uses a discount factor equal to one. The criterion is then sensitive to all increases in consumptions and treats generations equally. As was shown by Koopmans [27] and Diamond [5], no continuous preference ordering can be, at the same time, sensitive and treat generations equally but they do exist partial orderings which satisfy these axioms. The overtaking criterion is such an example. Gale [7] showed that, although it was not possible to compare all programs under that criterion, a partition of programs could be made into good or bad programs and that one could restrict himself to good programs. The concept of "optimal program" was first discussed. Various concepts of optimality were considered by Gale [7] and Brock [2] and "optimal programs" were shown to exist under various sets of assumptions about the technology and preferences. Then, on one hand, in order to relate the "undiscounted" case to the "discounted" case, Dana and Le Van [3],[4] introduced value functions for the overtaking criterion and showed that, under further hypotheses, an optimal program could be described as in the discounted case, by an optimal policy. On the other hand, non stationary versions of the overtaking criterion were used by Mckenzie [10] and more particularly by Michel[11] who characterized optimality by transversality conditions. We emphasize the fact that the literature on the overtaking criterion makes extensive use of price theory and turnpike results.

The chapter is organized as follows: In section 1.2, we set the model and show existence of a stationary optimal program and prices supporting it. In

section 1.3, we define good programmes and show that a partition of programs can be made into bad or good programs. Assuming strict concavity of the utility fonction at the stationary optimal program (respectively uniqueness of a stationary optimal program), we then show convergence (respectively average convergence) of good programs to the stationary optimal program. In section 1.4, we reconsider the two concepts of optimality ("optimality" and "weak optimality") introduced by Gale [7] and Brock [2] and show existence of optimal solutions for both concepts. We further characterize optimal solutions in terms of an Euler equation. In section 1.5, we introduce as in the discounted case, a value function and a Bellman's equation and on further assumptions on the technology and the criterion, show that an optimal program is unique and can be described by an optimal policy.

1.2 The Model

We consider an intertemporal economy where the instantaneous utility of the representative consumer depends on k_t, the capital stock on hand at date t and on k_{t+1}, the capital stock for date $t+1$. Given k_t, the set of feasible capital stocks for the next period $t+1$ is $\Gamma(k_t)$. We assume that at any period t, the feasible capital stock on hand belongs to X, a subset of $\mathbb{R}^n_+$. More explicitly, we make the following assumptions:

H1: X is a compact, convex set of $\mathbb{R}^n_+$ with non-empty interior and X contains 0.
H2: Γ is a continuous correspondence from X into X with non-empty convex images. Its graph, graph $\Gamma = \{(x,y) \in X \times X : y \in \Gamma(x)\}$, is convex.
H3: (Free disposal) If $y \in \Gamma(x), x' \geq x$ and $y' \leq y$, then $y' \in \Gamma(x')$.
H4: (Existence of expansible capital stocks) There exist $(x,y) \in$ graph Γ, with $y >> x$, i.e. $y_i > x_i$, for all $i = 1, ..., n$.
H5: The instantaneous utility function $F :$ graph $\Gamma \to \mathbb{R}$ is concave, continuous, increasing in first variable and decreasing in the second variable.

Remark 1.2.1. Assumption **H3** implies that $0 \in \Gamma(0)$. Assumptions **H3** and **H4** imply that the interior of graph Γ, denoted by int (graph Γ) is non-empty.

Definition 1.2.1. *A sequence* $\mathbf{x}$ *is feasible from* $x_0 \in X$ *if* $x_{t+1} \in \Gamma(x_t)$ *for all* $t \geq 0$. *A programme from* x_0 *is a feasible sequence from* x_0. *We denote by* $\Pi(x_0)$ *the set of feasible sequences from* x_0. *The set of programmes is denoted by* Π, *i.e.* $\Pi = \cup_{x \in X} \Pi(x)$.

We next define optimal stationary programmes and prove the existence of an optimal stationary programme and supporting prices.

Definition 1.2.2. *An optimal stationary programme is a solution* $\bar{x}$ *to the problem*

$$\max_{x \in X} \{F(x,x) : (x,x) \in \text{ graph } \Gamma\}$$