11. Theory of Stochastic Optimal Economic Growth

Lars J. Olson
University of Maryland, College Park, MD, USA

Santanu Roy
Southern Methodist University, Dallas, TX, USA

11.1 Introduction

Stochastic optimal growth involves the study of optimal intertemporal allocation of capital and consumption in an economy where production is subject to random disturbances. The theory traces its roots to the seminal work on deterministic optimal growth by Ramsey [108], Cass [21] and Koopmans [56]. Its influence has been enhanced by research that shows how the convex stochastic growth model can be decentralized to represent the behavior of consumers and firms in a dynamic competitive equilibrium of a productive economy ([104], [117], [15]). This makes the stochastic optimal growth model useful both as a normative exercise and in the development of positive theories of how the economy works. As a consequence, the theory has emerged as one of the central paradigms of dynamic economics. It is based on a simple, yet powerful model that encompasses fundamental questions that are basic to any theory of dynamic economic behavior: What are the characteristics and determinants of optimal policies? What are the economic incentives that govern the optimal intertemporal allocation of resources? What is the transient and long run behavior of variables in the model? Under different assumptions the model admits a rich set of answers to these questions.

Historically, the main focal point of the theory has been issues of aggregate economic growth. At the same time its primary variable, capital, has a flexible interpretation that allows the model and its extensions to represent a wide variety of economic problems ranging from the study of business cycles ([60], [64]) and asset pricing ([14], [15]) to the allocation of renewable natural resources ([78], [83], [84]). Equally important, the model provides a strong theoretical foundation for applied analysis of these problems. The model can be solved

numerically and has proved a testing ground for many numerical techniques used today in the analysis of dynamic economic problems.

This chapter provides an overview of key results in the *theory* of discounted stochastic optimal growth in discrete time.[1] The paper begins with an analysis of the classical stochastic growth model of Brock and Mirman [18] for a one-sector economy with a convex technology and utility that depends only on consumption.[2] We then consider extensions of the theory to problems with irreversible investment, increasing returns or a non-convex technology, experimentation and learning, and problems where utility depends on more than consumption alone. We develop the competitive price characterization of optimal policies that can be used to establish the equivalence between optimal and competitive outcomes; our focus, however, is on optimal solutions and their properties. The large literature on dynamic competitive equilibria is, therefore, left to the reader to explore. Likewise, we do not survey the many applications of the stochastic growth model. Instead, we focus on how the theory can be extended in different directions that have proved useful in application. Finally, we provide a glimpse of practical methods for solving the model, but the literature on numerical methods is too large for us to review here.

11.2 The Classical Framework

11.2.1 The One Sector Classical Model: Basic Properties

The stochastic growth model has three essential elements: an exogenous stochastic environment corresponding to random productivity disturbances, the production possibilities that determine the set of feasible allocations for consumption, investment and output, and an instantaneous welfare or utility function that represents the preferences of the agent or economic decision-maker. Productivity shocks at dates $t = 1, 2, ...,$ are denoted by $\{r_t\}$, a sequence of i.i.d. real-valued random variables, with common distribution ν on $B(\Phi)$, the Borel σ-field of $\Phi \subset \Re$. In particular, Φ is the support of ν and is assumed to be compact. Associated with this stochastic environment is a measure space $(\Omega, \mathcal{F}, \mu)$, where Ω is the set of all real sequences, $\mathcal{F}$ is the σ-field generated by cylinder sets of the form $\prod_{t=0}^{\infty} A_t$, where A_t belongs to $B(\Phi)$ for all t, and μ is the product distribution induced by ν. The statements: for a.e. ω and μ-a.s. mean "except for a subset of Ω of μ-measure zero". The random variable r_t is simply the t^{th} coordinate function on Ω. In the economy, output of a homogeneous consumption/capital good is produced via a production function that is homogeneous of degree one in capital and labor. This allows the economy to be

[1] There is a large literature on stochastic growth in continuous time that builds on Merton's [79] early work (see also, [16]).

[2] Previous surveys of stochastic growth such as [82] and [6] focus primarily on this case.