12. The von Neumann–Gale Growth Model and Its Stochastic Generalization

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12.1 Introduction

Von Neumann's [60] model of an expanding economy, generalized by Gale [30], was one of the first models in Mathematical Economics that served as the basis for a rich and interesting theory. This theory was developed for the most part in the 1950s and 1960s. Substantial contributions to it were made by such outstanding economists and mathematicians as McKenzie, Radner, Rockafellar, Nikaido, Morishima and others (see, e.g., the monograph by Nikaido [47] and references therein).

The theory of the von Neumann–Gale model, in its classical form, is purely deterministic. It does not reflect the influence of random factors on economic growth. The importance of taking these factors into account was realized early on. First attempts aimed at the construction of stochastic analogues of the von Neumann–Gale model were undertaken in the 1970s by Radner [51, 52]. However, the initial attack on the problem left many questions unanswered. Studies in this direction faced serious mathematical difficulties. To overcome these difficulties, new mathematical techniques were required, that were developed only during the last decade. The main purpose of the present paper is to provide an account of recent achievements in the field.

Along with probabilistic generalizations of the classical results, new applications of the stochastic version of the von Neumann–Gale model will be highlighted: the applications to the analysis of the dynamics of financial markets.

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This marks an unexpected change in focus, as well as a revival of the model because the framework—that originally aimed at the modeling of economic growth—turned out to be very natural in connection with financial issues. The financial aspects pose a number of interesting new questions, which are currently only partially answered and which indicate directions for further work.

The basic mathematical framework of the von Neumann–Gale model is that of set-valued dynamical systems, e.g. Akin [2]. Dynamics of such systems are described by multivalued operators specifying for every state of the system "today" a set of possible states "tomorrow." The characteristic features of the operators associated with the von Neumann–Gale model are certain properties of convexity and homogeneity. A profound mathematical study of such dynamical systems has been carried out by Rockafellar [54], Makarov and Rubinov [41] and others. In the stochastic case, one has to deal with *random* set-valued dynamical systems possessing analogous properties of convexity and homogeneity. For an introduction to the theory of random dynamical systems see Arnold [8].

In the theory of economic growth, the von Neumann–Gale model occupies quite a special position. By and large, models of economic growth prevailing in the current literature belong to one of the following two types. They either assume that the growth rates of economic factors (e.g. labor) are given exogenously, or consider the phenomenon of economic growth from the point of view of endogenous changes in the technology or production functions. Examples of models of the former kind are those proposed by Solow [56] and Ramsey [53]. The analysis of process of growth in such models consists essentially in the study of optimal *proportions* of growth (e.g. expressed in terms of *per capita* quantities). In this context, paths of the system can exogenously be normalized, after which the work basically reduces to the analysis of optimization problems with bounded state variables. The stochastic theory of such models is well developed. Foundations for it were laid in the 1970s and 1980s in the work of Dynkin, Radner, Brock, Mirman, Bewley, Dana, Majumdar, Mitra, Zilcha and others. Results obtained in this field have been reflected in a number of surveys and monographs, e.g. Mirman [44], Dynkin and Yushkevich [19], Arkin and Evstigneev [7], Stokey, Lucas and Prescott [59], Brock and Dechert [13], Olson and Roy [49], containing references to the original papers. For more recent work, see Amir [4], Amir and Evstigneev [5], Mitra, Montrucchio and Privileggi [45] and Stachurski [58].

Models of the second kind—in which endogenous changes in the production function or technology are analyzed—are systematically reviewed, for example, in Aghion and Howitt [1] and Barro and Sala-i-Martin [11]. The theory of stochastic endogenous growth models is still in its infancy. Results in this direction have been obtained by de Hek [14] and de Hek and Roy [15].

The von Neumann–Gale model does not fit into either of the above two classes of models. The "technology" in it is given exogenously, and in the stationary deterministic case, it does not change in time. However, the growth rates are endogenous: they are derived from the model itself. The maximum growth rate (the von Neumann rate) is a solution to an optimization problem