2. Optimal Growth Models with Discounted Return

Cuong Le Van
CERMSEM, CNRS, University Paris 1, France[1]

In this chapter, we provide a unified treatment of a class of optimal growth models by using dynamic programming methods. In the economies we consider in this chapter, a social planner maximizes a discounted sum of utilities which depend on the current and past period states subject to a feasibility constraint. We show that this problem can be brought down to a sequence of static problems by using the value function of the problem and the associated Bellman equation. The Bellman equation allows us to state that
(i)the value function is continuous with respect to the initial data and to the discount factor,
(ii) the optimal trajectory of state variables can be described as a dynamical system (which may be multi-valued)
We first give two examples of optimal growth models.
Example 1
Consider a two-sector economy. At date t, sector 1 produces consumption good c_t by using a capital stock k_t^1 which is produced in sector 2. At date t, sector 2 produces capital stock k_{t+1} which will be used in period $t+1$ by the two sectors. To produce k_{t+1}, sector 2 needs a quantity k_t^2 of capital good. The social planner solves at date 0 the following problem:

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t), \beta \in]0,1[,$$

under the constraints:

$$\forall t,\ 0 \le c_t \le f^c(k_t^1),$$
$$0 \le k_{t+1} \le f^k(k_t^2),$$

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$$k_t^1 + k_t^2 \leq k_t$$
$$k_t^1 \geq 0, k_t^2 \geq 0,$$

and $k_0 \geq 0$ is given. The functions f^c and f^k are respectively the production functions of the consumption good sector and of the capital good sector.
The reader can check that, if the utility function u and the production functions f^c and f^k are strictly increasing, the initial problem becomes:

$$\max \sum_{t=0}^{+\infty} \beta^t V(k_t, k_{t+1}), 0 < \beta < 1,$$

under the constraints:

$$\forall t \geq 0,\ \ k_{t+1} \in \Gamma(k_t),$$

and $k_0 \geq 0$ is given. The correspondence Γ is defined by $\forall k \geq 0, \Gamma(k) = [0, f^k(k)]$, and the return function V by $V(k_t, k_{t+1}) = u(f^c(k_t - (f^k)^{-1}(k_{t+1})))$, the function $(f^k)^{-1}$ being the inverse function of f^k.
Example 2 (Human capital; Lucas [10]).
We have an one-sector growth model. But the output is a function of physical capital k and of effective labor N^e. Effective labor is the sum of skill-weighted manhours devoted to current production. More explicitly, assume there are N identical workers. Each worker has $h \in [0, +\infty[$ as skill level and devotes a fraction θ of his non-leisure time to current production and the remaining $(1-\theta)$ to human capital accumulation. We thus have $N^e = Nh\theta$. Given k, h, θ, N, the level of output is $Ah^\gamma F(k, Nh\theta)$. The total productivity now is Ah^γ. The term h^γ captures the external effects of human capital while the technology level A is assumed to be constant.
We assume that the rate of growth of human capital depends, through a function G, on the non-leisure time devoted to its accumulation.
The model is as follows:

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t), 0 < \beta < 1,$$

under the constraints:

$$\forall t \geq 0, c_t + k_{t+1} - (1-\delta)k_t \leq Ah_t^\gamma F(k_t, Nh_t\theta_t),$$
$$h_{t+1} \leq h_t(1 + G(1-\theta_t))$$
$$h_t \geq 0, k_t \geq 0,$$

and $k_0 \geq 0, h_0 \geq 0$ are given.

Let $x = (k_0, h_0) \in \mathbb{R}_+^2,\ y = (k_1, h_1) \in \mathbb{R}_+^2$. Define the indirect utility V:

$$V(x, y) = \max_{c,\theta}\{u(c)\}$$

under the constraints: