3. Duality Theory in Infinite Horizon Optimization Models

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3.1 Introduction

In intertemporal resource allocation problems with no terminal date, price systems which characterize efficient or optimal allocations have figured prominently since the pioneering contribution by Malinvaud (1953). The method of duality theory that has been developed to study such problems relies on convex analysis and may be viewed as an extension of the corresponding theory for static or finite horizon allocation problems. The purpose of this survey is to introduce the reader to this method by showing how it has been applied in the literature dealing with optimal intertemporal allocation, when future utilities are discounted, which constitutes only a part (although a significant one) of the class of problems referred to above.

A major accomplishment of this literature is the result that, in a very general framework of capital accumulation (often referred to in the literature as a reduced-form model), optimal programs may be characterized by the existence of dual variables, interpreted as "shadow prices", such that at these prices the given program satisfies the so-called "competitive conditions" and the "transversality condition". The competitive conditions are analogous to those in static or finite horizon optimality problems, and involve myopic (generalized) intertemporal profit maximization. The fundamental difference stems from the infinite-horizon nature of the problem, and is captured by the transversality condition.

The usefulness of this central result may be described as follows. Sufficient conditions (in terms of shadow prices) for a program to be optimal can be used to check whether a candidate program is optimal, if one has a good idea of

[1] Discussion over the years with many persons has influenced my understanding of the subject matter covered in this essay. They include David Cass, Swapan Dasgupta, Ali Khan, Mukul Majumdar, Lionel McKenzie, Kazuo Nishimura, Bezalel Peleg, Debraj Ray and Itzhak Zilcha.

shadow prices that support such a program. This makes duality theory a principal alternative to dynamic programming methods in solving for an optimal program. Necessary conditions (in terms of shadow prices) for a program to be optimal can be used to obtain qualitative properties of an optimal program without necessarily solving for an optimal program.

Even though the theory of optimal growth dates back to the seminal contribution of Ramsey (1928), versions of the "price chracterization result", referred to above, were developed almost forty years later, in the papers of Gale (1967), McFadden (1967) and McKenzie (1968). Following Ramsey's lead, the principal concern of these papers was the theory of undiscounted optimal growth in general capital accumulation models. Subsequently, methods of duality theory were applied to the discounted case by Peleg (1970) and Peleg and Ryder (1972). However, it is only with the contribution of Weitzman (1973) that we have a completely satisfactory price characterization result for the discounted case. The setting for his result is a very general and flexible framework of capital accumulation (described here in Section 3.2), and his approach (combining elements of duality theory and dynamic programming) makes the logic of the result (and the assumptions needed for its validity) entirely transparent. We present the basic characterization result, following his approach, in Section 3.3.

Dual variables have been used very effectively in the literature on optimal intertemporal allocation in obtaining another major result, namely the existence of a non-trivial stationary optimal program, supported by "quasi-stationary" shadow prices. Versions of this result appear in Sutherland (1970) and Peleg and Ryder (1974). But, for the general framework described in Section 3.2, the result was developed later by Flynn (1980) and McKenzie (1982). The approach used in these two papers is to establish the existence of a discounted golden-rule (analogous to a golden-rule in the undiscounted case) by a fixed point argument, and then support this discounted golden-rule by appropriate dual variables. We present this theory in Section 3.4.

The fact that there exists a stationary optimal program with quasi-stationary price support allows one to revisit the basic price characterization result (of Section 3.3), and develop an alternative version of it which helps to identify non-optimal competitive programs in a finite number of periods. The transversality condition is an asymptotic condition, and can never be verified in finite time. It turns out that a convenient period-by-period condition can replace the transversality condition in the price characterization theorems, and so a violation of this condition in any period immediately signals non-optimality. Such a period-by-period condition was first proposed and established by Brock and Majumdar (1988) in the undiscounted case, and the theory for the discounted case was developed subsequently in Dasgupta and Mitra (1988). We present this theory in Section 3.5.

Although the transversality condition is both necessary and sufficient for optimality of competitive programs, there is a fairly wide and interesting class of models in which the competitive conditions alone are sufficient to ensure optimality, and the transversality condition is superfluous. That is, programs