4. Rationalizability in Optimal Growth Theory

Gerhard Sorger
Department of Economics, University of Vienna, Austria

4.1 Introduction

Before I address rationalizability in optimal growth theory, let me discuss the issue of rationalizability in more general terms. Suppose there exists a class of models, $\mathcal{M}$, such that every model $M \in \mathcal{M}$ describes the economic phenomenon under consideration. Suppose furthermore that, for every $M \in \mathcal{M}$, there exists a (possibly empty) set of 'solutions' or 'equilibria' $H(M) \subseteq \mathcal{H}$, where $\mathcal{H}$ is a fixed set of possible solutions. One has to distinguish between four different problems related to $M \in \mathcal{M}$ and $h \in \mathcal{H}$, respectively.

- The *solution problem* for the model M consists in finding or characterizing one or all elements of $H(M)$.
- The *existence problem* for the model M consists in determining whether $H(M)$ is empty or not.
- The *inverse problem* for the solution h consists in finding or characterizing one or all models $M \in \mathcal{M}$ such that $h \in H(M)$.
- The *rationalizability problem* for the solution h consists in determining whether there exists a model $M \in \mathcal{M}$ such that $h \in H(M)$.

From the above descriptions one can see that the problem of rationalizing a given solution h is related to the inverse problem for h in the same way as the existence problem for the model M is related to the problem of solving M. It is furthermore obvious that every solution h that can be rationalized by the class $\mathcal{M}$ can also be rationalized by any class of models that includes $\mathcal{M}$. Rationalizability of a given solution h by a restricted class of models is therefore a stronger property than rationalizability by a wider class of models. Questions regarding rationalizability have been discussed in a number of different branches of economic theory.[1] An early discussion of the inverse problem in optimal growth theory is the paper by Kurz [7]; see also Chang [3]

[1] A prominent example is the Sonnenschein-Mantel-Debreu theorem. It says that every continuous function, which is homogeneous of degree 0 and satisfies Walras'

and references therein. The issue of rationalizability in optimal growth theory received a strong impetus in the early to mid 1980s, when it was first shown that dynamic macroeconomic models satisfying standard assumptions can generate complicated deterministic dynamics. In other words, this line of research demonstrated the rationalizability of chaotic dynamical systems by certain classes of intertemporal macroeconomic models. For example, in an often cited paper, Boldrin and Montrucchio [1] proved that every twice continuously differentiable function can be the optimal policy function of an infinitely-lived agent model in reduced form, which has the smoothness and convexity properties that are typically assumed by growth theorists. A consequence of this result is that even the most complicated dynamic behavior cannot be ruled out by the standard assumptions of optimal growth theory. It was furthermore argued that the aperiodic fluctuations generated by chaotic dynamical systems can resemble realistic business cycles and that standard optimal growth models are therefore not only consistent with, but can actually explain important stylized facts of the business cycle.[2] However, it soon became clear that the constructive approach used by Boldrin and Montrucchio [1] depends on the choice of unrealistically high rates of time-preference. Sorger [27] provided the first rigorous proof that high time-preference is indeed necessary for the rationalizability of complicated dynamics by optimal growth models or, in other words, that there exist non-trivial discount factor restrictions for the optimality of complicated dynamics. In the present chapter, I survey the literature that has emerged from Boldrin and Montrucchio's and from Sorger's contributions and that addresses the questions of rationalizability and discount factor restrictions in optimal growth models. The rest of this chapter is organized as follows. In section 4.2, I specify three different classes of optimal growth models and I state a few important results about the solutions of such models. This allows me to give a precise definition of the rationalizability problem in optimal growth theory. Section 4.3 summarizes necessary and sufficient conditions for the rationalizability of given functions as optimal policy functions of infinitely-lived agent models. Typically, these conditions are formulated in terms of smoothness properties of the optimal policy functions. For the most comprehensive of the three classes of optimal growth models, I show that every rationalizable function is necessarily continuous and that every Lipschitz-continuous function can be rationalized. Moreover, I illustrate by means of examples that closing the gap between these two conditions is likely to be very difficult. Section 4.4 reviews the literature on discount factor restrictions for the rationalizability of complicated dynamics. The first main result is a simple relation between the discount factor of an optimal growth model and the topological entropy of the

law, is the excess demand function of a static competitive economy with standard properties; see Sonnenschein [24, 25], Mantel [10], and Debreu [4]. In this setting, a model M is described by the number of households, their preferences, and their endowments, and the solution of M is the excess demand function generated by M.

[2] See, e.g., Boldrin and Woodford [2] for a critical discussion of these arguments.