5. On Stationary Optimal Stocks in Optimal Growth Theory: Existence and Uniqueness Results

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5.1 Introduction

The concept of a non-trivial stationary optimal stock (SOS) plays a central role in the theory of optimal intertemporal allocation over an infinite horizon. While the optimal policy correspondence describes fully optimal behavior in such models, it is quite difficult to compute it accurately, and it can be solved in explicit form in only a very few highly specialized examples.

However, if non-stationary optimal programs, after a period of transition, are close to a certain stationary program (and the transition period is not very long), then their behavior can be approximately described by the stationary optimal program. Thus, even though it is only by accident that an economy has exactly a stationary optimal stock as its initial stock, a study of the existence, uniqueness and (local and global) stability of stationary optimal programs is of considerable significance.

Furthermore, if one is interested in comparative dynamics in this framework, one observes that it might be very difficult to get definitive results for policy purposes by varying a parameter and seeing the effect of it on the entire optimal policy correspondence. On the other hand, if the stationary optimal program is at least locally stable, then one can often predict the change in the stationary optimal program following a "small" change in a parameter, and this can enable one to conduct local comparative dynamics exercises in this framework.

In this essay, we present the basic results on the existence and uniqueness of (non-trivial) stationary optimal programs. A comprehensive account of the

[1] Our intellectual debt to William Brock and Lionel McKenzie, for our understanding of the subject matter of this survey, should be quite obvious. In writing this survey, we have relied heavily on our collaborative research with Jess Benhabib, Swapan Dasgupta, and Ali Khan.

stability (or turnpike) property of stationary optimal programs is already available in McKenzie (1986), and we refer the reader to his definitive study of this topic.

The existence of a stationary optimal stock (briefly, SOS) in multi-sector optimal growth models has been shown by Sutherland (1970) Hansen and Koopmans (1972), Peleg and Ryder (1974), Cass and Shell (1976), Flynn (1980), McKenzie (1982, 1986) and Khan and Mitra (1986), among others. We follow very closely the approach in Khan and Mitra (1986).

The demonstration of existence typically consists of three separate steps. First, a fixed point argument is used to show the existence of what we call in the sequel, a discounted golden-rule stock. Second, a separation argument in the form of the Kuhn-Tucker theorem is used to provide a "price-support" to the discounted golden-rule stock. Finally, a computation based on the price support property is used to show that the discounted golden-rule stock is optimal among all programs starting from that stock.

This approach, relying on duality theory (in the second and third steps), is followed by Peleg and Ryder (1974), Cass and Shell (1976), Flynn (1980), McKenzie (1982, 1986). An exception to this is Sutherland (1970) who relies on methods of dynamic programming and is able to avoid supporting prices and the Kuhn-Tucker theorem. However, Sutherland does not establish the existence of a *non-trivial* SOS, and as noted by Peleg and Ryder (1974), the null stock is always a SOS in a set-up which allows for the possibility of inaction, and does not allow production of positive outputs from zero inputs.

Khan and Mitra (1986) use a purely primal approach to the existence of a non-trivial SOS, and by a simple computation based on Jensen's inequality, establish that a discounted golden-rule stock is always a SOS. Thus, once the fixed point argument (the first step in the three-step argument indicated above) ensures the existence of a discounted golden-rule stock, the existence of a stationary optimal stock is also assured. This primal approach does not suffer from the shortcoming noted in the dynamic programming method, for it is simple to identify a condition on the economy (known as $\delta - normality$) which ensures that the discounted golden-rule stock (and therefore the corresponding stationary optimal program) is non-trivial.

The existence of a discounted golden-rule stock therefore emerges as a key concept of this subject. The idea is to approach an infinite-horizon optimization problem by solving an appropriate two-period optimization problem.

A direct payoff of the primal approach of Khan and Mitra (1986) is that an assumption frequently used in this literature (known as $\delta - productivity$) can be dispensed with, since its role is simply to ensure that Slater's condition holds when one invokes the Kuhn-Tucker theorem (in the second step of the three-step argument).

Following Khan and Mitra (1986), we also use a purely primal approach to show that a SOS, k, is always a discounted golden-rule stock, provided (k, k) is in the interior of the technology set. This result is proved by McKenzie (1986), relying on duality methods. Again, the proof involves three steps. First, a