6. Optimal Cycles and Chaos

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6.1 Introduction

Optimal growth models have originally been developed in order to analyze the long-run implications of capital accumulation and technological progress. Later on, however, it has been noticed that essentially the same model structure can also be used to shed light on short-run phenomena like the business cycle. The most prominent outcome of this line of research are real-business-cycle (RBC) theories, which assume that business cycles are triggered by exogenous stochastic shocks and which analyze the mechanisms by which these shocks propagate through the economy. The literature surveyed in the present chapter, on the other hand, shows that optimal growth models can generate business cycles even in the absence of exogenous shocks. It is therefore appropriate to refer to these results as endogenous-business-cycle (EBC) theories. An important property common to both RBC and EBC theories is that the business cycles qualify as optimal programs. In other words, the solutions of both RBC and EBC models are Pareto-efficient. As far as the deterministic EBC models are concerned, the most important implication of this fact is that the standard assumptions of optimal growth theory do not rule out intrinsic instability of the economy, an instability that allows for periodic or even chaotic optimal programs.

Following Ramsey (1928), much of the earlier literature in optimal growth theory focused on equal treatment of generations over time, and therefore on

the undiscounted case. The analysis of this class of models was brought to maturity in the papers of Gale (1967), McKenzie (1968), and Brock (1970). The treatment of the case in which future utilities were discounted was typically done in the one-sector neoclassical model, where the significant difference between the two cases was not revealed because of the one-to-one conversion of capital to consumption good implicit in its formulation. It was with the examples of Kurz (1968) and Sutherland (1970), in models which did not have this feature, that one recognized that discounting the future in general provided more limited intertemporal arbitrage opportunities; thus, the standard argument for smoothing out cyclical behavior was not valid in such frameworks.

Samuelson (1973) can be considered to provide definitive directions for research towards an understanding of such a phenomenon. On the one hand, he reported an example, due to Weitzman, which showed that cyclical optimal behavior was consistent with interior solutions to Ramsey-Euler equations and therefore would not disappear with assumptions which ruled out boundary solutions to optimal growth problems. On the other hand, he conjectured that, if the utility function was strictly concave, then cyclical optimal behavior of the Weitzman type would not arise, if the planner was sufficiently patient. The second idea was formalized in terms of turnpike theorems for low discount rates in a *Journal of Economic Theory* symposium of 1976, and led to a literature which is comprehensively surveyed in McKenzie (1986). The first idea led Benhabib and Nishimura (1985) to initiate a systematic investigation of the sources of optimal cycles and this, in turn, led to the literature that is surveyed in this chapter.

Section 6.2 sets the stage for our survey by presenting background material on dynamical systems and optimal growth models. Sections 6.3 and 6.4 form the main part of the chapter. In section 6.3 we study the optimality of periodic cycles. Although periodic optimal growth paths cannot be interpreted as realistic business cycles, the characterization of the conditions under which periodic cycles are optimal allows important insights into the mechanisms that can generate non-monotonic optimal growth paths. Section 6.4 then turns to chaotic optimal growth paths. These solutions resemble actual business cycles more closely than periodic ones, but it is somewhat harder to characterize the mechanisms by which they are generated.

6.2 Basic Definitions and Results

This section presents some background material that is necessary to state the main results on optimal cycles and chaos. First we discuss a number of concepts and results that are related to cyclical and chaotic behavior of dynamical systems. Then we formulate the reduced form optimal growth model and show that it encompasses, among other models, a discrete-time version of the two-sector model of Uzawa (1964).