2

Preliminaries

This chapter provides some mathematical background and basic notions concerning binary relations, partial orders, monoids, rational and recognizable languages, and Turing machines.

2.1 Words and Partial Orders

Let Σ be an *alphabet*, i.e., a nonempty finite set of *symbols* or *letters*. The set of (finite) *strings* or *words* over Σ is denoted by Σ^*, the set of nonempty words by Σ^+. The empty string, i.e., the neutral element with respect to word concatenation, is denoted by ε. Note that words will later be seen as system runs and therefore be modeled as a special case of graphs. This is, however, just to treat the objects that represent the behavior of a distributed system in a unified manner and does not affect the notions introduced so far. Thus, we may, throughout the book, consider a word to be a sequence of letters.

Given a set V and binary relations $\mathcal{R}_1, \mathcal{R}_2 \subseteq V \times V$, the *product* $\mathcal{R}_1 \circ \mathcal{R}_2 \subseteq V \times V$ of $\mathcal{R}_1$ and $\mathcal{R}_2$ is the binary relation $\{(u,v) \in V \times V \mid \text{there is } w \in V \text{ such that } (u,w) \in \mathcal{R}_1 \text{ and } (w,v) \in \mathcal{R}_2\}$. Given $\mathcal{R} \subseteq V \times V$, we moreover define $\mathcal{R}^0$ to be $\{(u,u) \mid u \in V\}$ and $\mathcal{R}^{n+1}$ to be $\mathcal{R} \circ \mathcal{R}^n$ for any $n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers including 0). Finally, let $\mathcal{R}^*$ denote the infinite union $\bigcup_{n \in \mathbb{N}} \mathcal{R}^n$. Note that, instead of $(u,v) \in \mathcal{R}$, we may also write $u\mathcal{R}v$.

A binary relation $\leq \subseteq V \times V$ on a set V is called

- *reflexive* if, for each $u \in V$, $u \leq u$,
- *irreflexive* if, for each $u \in V$, $u \not\leq u$,
- *transitive* if, for any $u, v, w \in V$, $(u \leq v \wedge v \leq w)$ implies $u \leq w$, and
- *antisymmetric* if, for any $u, v \in V$, $(u \leq v \wedge v \leq u)$ implies $u = v$.

As mentioned in the introduction, one single behavior of a distributed system will be described in a compact manner by a partially ordered set.

Definition 2.1 (Partially Ordered Set). *A* (finite) partially ordered set *(also called a* poset*) is a pair* $(V, \leq)$ *such that*

- V *is a finite set, and*
- $\leq$ *is a binary relation on* V *that is reflexive, transitive, and antisymmetric.*

Given a poset $(V, \leq)$, we call the relation $\leq$ a *partial order*. A *totally ordered set* is a poset $(V, \leq)$ such that, for any $u, v \in V$, $u \leq v$ or $v \leq u$. Accordingly, we then call the relation $\leq$ a *total order*.

Let $\mathcal{P} = (V, \leq)$ be a poset. By $<$, we denote the binary relation $\leq \setminus \{(u, u) \mid u \in V\}$. Moreover, for $u, v \in V$, let us write $u \lessdot v$ if both $u < v$ and, for any $w \in V$, $u < w \leq v$ implies $w = v$. Then, $(V, \lessdot)$ and $\lessdot$ are called the *Hasse diagram* of $\mathcal{P}$ and, respectively, the *covering relation* of $\leq$. For $u \in V$, we furthermore say that u is *minimal*/*maximal* in $\mathcal{P}$ (we also say minimal/maximal in $(V, <)$) if there is no $v \in V$ such that $v < u$/$u < v$, respectively. Given an alphabet Σ, a *Σ-labeled poset* is a triple $(V, \leq, \lambda)$ such that $(V, \leq)$ is a poset and λ is a function $V \to \Sigma$, called a *labeling function*.

Throughout this book, we do not distinguish isomorphic structures.

2.2 Monoids and Languages

The objects considered when modeling the behavior of a distributed system are often equipped with a concatenation function, which allows us to combine single behaviors towards more complex ones. Together with a unit element, this gives rise to a monoid.

Definition 2.2 (Monoid). *A* monoid *is a triple* $(\mathbb{M}, \cdot, \mathbf{1})$ *such that*

- $\mathbb{M}$ *is a nonempty set,*
- $\_ \cdot \_$ *is an associative mapping* $\mathbb{M} \times \mathbb{M} \to \mathbb{M}$ *(i.e.,* $(t_1 \cdot t_2) \cdot t_3 = t_1 \cdot (t_2 \cdot t_3)$ *for any* $t_1, t_2, t_3 \in \mathbb{M}$*), and*
- $\mathbf{1} \in \mathbb{M}$ *is the* unit *satisfying* $\mathbf{1} \cdot t = t \cdot \mathbf{1} = t$ *for any* $t \in \mathbb{M}$.

A monoid $(\mathbb{M}, \cdot, \mathbf{1})$ is often identified with its universe $\mathbb{M}$. A subset of $\mathbb{M}$ is called a *language*. Given languages $L_1, L_2 \subseteq \mathbb{M}$, the *product* of L_1 and L_2 is denoted by $L_1 \cdot L_2$ and defined to be the set $\{t_1 \cdot t_2 \mid t_1 \in L_1 \text{ and } t_2 \in L_2\}$. Furthermore, we set $L^0 := \{\mathbf{1}\}$ and, for $n \in \mathbb{N}$, $L^{n+1} := L \cdot L^n$. The *iteration* of L is defined to be $L^* := \bigcup_{n \in \mathbb{N}} L^n$, which is also denoted by $\langle L \rangle_{\mathbb{M}}$. Note that $\langle L \rangle_{\mathbb{M}}$ is a submonoid of $\mathbb{M}$. By L^+, we abbreviate $\bigcup_{n \in \mathbb{N}_{\geq 1}} L^n$ where $\mathbb{N}_{\geq 1}$ will throughout this book stand for the set of positive natural numbers. A language $L \subseteq \mathbb{M}$ is called *finitely generated* if there is a finite subset Π of $\mathbb{M}$ such that $L \subseteq \langle \Pi \rangle_{\mathbb{M}}$. In that case, we also say that L is *finitely generated by Π*.

Definition 2.3 (Rational Language). *Let* $(\mathbb{M}, \cdot, \mathbf{1})$ *be a monoid. The class* $\mathcal{RAT}_{\mathbb{M}}$ *of* rational *subsets of* $\mathbb{M}$ *is the least set* R *satisfying*