6

Mazurkiewicz Traces and Asynchronous Automata

We will now study Mazurkiewicz traces, which have been touched on in the previous chapter, in more detail. Mazurkiewicz traces are suitable to model communication where components synchronize by executing certain actions simultaneously, whereas others are taken autonomously.

6.1 Mazurkiewicz Traces

Recall that Mazurkiewicz traces are classified as M^+-traces or M^--traces:

Definition 6.1 (M^+-Trace). *An* M^+-trace *over* $\widetilde{\Sigma}$ *is a dag* $(V, \{\lhd_\ell\}_{\ell \in 2^{Ag}}, \lambda)$ *from* $\mathbb{DAG}(\widetilde{\Sigma}, 2^{Ag})$ *such that*

- $\lhd = \bigcup_{i \in Ag} \lessdot_i$*, and*
- *for any* $(u, v) \in \lhd$ *and* $\ell \in 2^{Ag}$*,* $u \lhd_\ell v$ *iff* $\ell = \{i \in Ag \mid u \lessdot_i v\}$*.*

Definition 6.2 (M^--Trace). *An* M^--trace *over* $\widetilde{\Sigma}$ *is a* $\widetilde{\Sigma}$*-dag* $(V, \lhd, \lambda) \in \mathbb{DAG}_H(\widetilde{\Sigma})$ *such that, for any* $u, v \in V$*,* $u \lhd v$ *implies* $\lambda(u)\ D_{\widetilde{\Sigma}}\ \lambda(v)$*.*

The set of M^+-traces over $\widetilde{\Sigma}$ is denoted by $\mathbb{TR}^+(\widetilde{\Sigma})$, the set of M^--traces over $\widetilde{\Sigma}$ by $\mathbb{TR}^-(\widetilde{\Sigma})$. As usual, we often write $\mathbb{TR}^+$ (not to be confused with the product operation) or $\mathbb{TR}^-$ if $\widetilde{\Sigma}$ can be learned from the context. Observe that $\mathbb{TR}^+(\widetilde{\Sigma})$ might be seen as a subset of $\mathbb{DAG}(\widetilde{\Sigma}, 2^{Ag} \setminus \{\emptyset\})$. Moreover, $\mathbb{TR}^-(\widetilde{\Sigma}) \subseteq \mathbb{DAG}_{\rightrightarrows}(\widetilde{\Sigma})$, whereas, in general, $\mathbb{TR}^+(\widetilde{\Sigma}) \subseteq \mathbb{DAG}_{\rightrightarrows}(\widetilde{\Sigma}, 2^{Ag})$ does not hold. Remarkably, if $D_{\widetilde{\Sigma}} = \Sigma \times \Sigma$, then any trace (no matter if M^+or M^-) constitutes a totally ordered set and we even have $\mathbb{TR}^-(\widetilde{\Sigma}) \subseteq \mathbb{W}(\Sigma)$.

Whenever we leave open whether we deal with an M^+- or an M^--trace over $\widetilde{\Sigma}$, we write $(V, \prec, \lambda)$ to refer to either $(V, \{\lhd_\ell\}_{\ell \in 2^{Ag}}, \lambda)$ or $(V, \lhd, \lambda)$, respectively. Accordingly, $\prec'$ is actually meant to be either a collection of relations $\{\lhd'_\ell\}_{\ell \in 2^{Ag}}$ or a relation $\lhd'$.

Example 6.3. Recall that part (a) of Fig. 5.4 on page 48 depicts an M^+- trace over the distributed alphabet $(\{a,b,d\},\{a,b,e\},\{a,b\})$, whereas Fig. 5.4b shows an M^--trace. Moreover, Fig. 6.1a and Fig. 6.1b depict an M^+- and an M^--trace over $(\{a,b\},\{b,c\},\{c\})$, respectively (say, with $Ag = \{1,2,3\}$).

Usually, Mazurkiewicz traces are defined as labeled posets $(V, \leq, \lambda)$ [27]. More specifically, we call a Σ-labeled poset $(V, \leq, \lambda)$ a *poset-trace* over $\widetilde{\Sigma}$ if

- for any $u, v \in V$, if $\lambda(u)\ D_{\widetilde{\Sigma}}\ \lambda(v)$, then $u \leq v$ or $v \leq u$, and
- for any $u, v \in V$, $u \lessdot v$ implies $\lambda(u)\ D_{\widetilde{\Sigma}}\ \lambda(v)$.

But to treat all the structures relevant to this book in a common framework, a trace is given by its graphical representations. However, there is a one-to-one correspondence of poset-, M^+-, and M^--traces: for any $\alpha \in \{+,-\}$ and any poset-trace $\mathcal{P}$ over $\widetilde{\Sigma}$, there is a unique M^α-trace $\mathcal{T} = (V, \prec, \lambda) \in \mathbb{TR}^\alpha(\widetilde{\Sigma})$ (recall that $\prec$ is meant to be either $\{\lhd_\ell\}_{\ell \in 2^{Ag}}$ or $\lhd$) such that $(V, \leq, \lambda) = \mathcal{P}$. It is therefore justified to call $\mathcal{T}$ *the* M^α*-trace* of $\mathcal{P}$. Moreover, we can state that, for any given word $\underline{w} \in \mathbb{W}(\Sigma)$, there is exactly one M^α-trace $\mathcal{T}$ such that $\underline{w} \in Lin(\mathcal{T})$. For example, Fig. 6.1 depicts the only M^+-trace/M^--trace over $(\{a,b\},\{b,c\},\{c\})$ with linearization *cbacbb*. Thus, the traces from Fig. 6.1 constitute two different views of one and the same behavior.

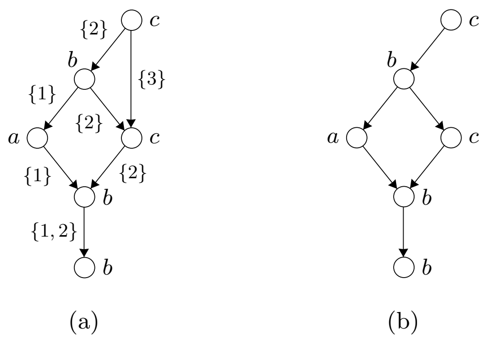

Fig. 6.1. An M^+-trace and an M^--trace over $(\{a,b\},\{b,c\},\{c\})$

6.2 Trace Languages

For this section, we fix $\alpha \in \{+,-\}$. For M^α-traces $\mathcal{T} = (V, \prec, \lambda)$ and $\mathcal{T}' = (V', \prec', \lambda')$ over $\tilde{\Sigma}$, let $\mathcal{T} \cdot \mathcal{T}'$ denote the *concatenation* of $\mathcal{T}$ and $\mathcal{T}'$, which is the M^α-trace of the poset-trace $(V'', \leq'', \lambda'')$ over $\widetilde{\Sigma}$ with $V'' = V \uplus V'$, $\lambda'' = \lambda \uplus \lambda'$, and $\leq'' = (\lhd \uplus \lhd' \uplus \{(u,v) \in V \times V' \mid (\lambda(u), \lambda'(v)) \in D_{\widetilde{\Sigma}}\})^*$. It is an easy task to show that trace concatenation is associative.