Limit Cycles in Quantum Mechanics

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1 Introduction

This lecture concerns limit cycles in renormalization group (RG) behavior of quantum Hamiltonians. Cyclic behavior is perhaps more common in quantum mechanics than the fixed-point behavior which is well-known from critical phenomena in classical statistical mechanics. We discuss a simple Hamiltonian model that exhibits limit cycle behavior.

The value of the model is its simplicity. The model can serve as a basis for further mathematical studies of the cycle for the purpose of solving complex physical theories. For example, the limit cycle may be related to existence of a set of bound states with binding energies forming a geometric sequence converging to zero. The model offers a possibility of studying in detail how Hamiltonians with bound states behave as operators under the RG transformation in the vicinity of the cycle. This lecture reports on results concerning behavior of marginal and irrelevant operators in the model.

The possibility of limit cycle behavior in RG calculations was originally pointed out by Wilson [1] in the context of theory of strong interactions in particle physics. The classification of operators near a fixed point as relevant, marginal, and irrelevant, was introduced by Wegner [2] in the context of critical phenomena and corrections to scaling in condensed matter physics.

Recent interest in the RG limit cycle stems from the structure of three-nucleon bound-state spectrum in the case of two-body nucleon-nucleon interactions that have a very short range in comparison to the scattering length. This structure was first noticed by Thomas [3] and subsequently discussed by Efimov [4]. It was only recently associated with ultraviolet regularization dependence of parameters in the nuclear potentials by Bedaque, Hammer, and van Kolck [5]. Interactions of atoms, especially in Helium trimers, may exhibit a similar cycle structure even more transparently than effective nuclear forces because more bound states may exist for atoms than for nuclei. In addition, atomic interactions are easier to tune close to the cycle structure by changing experimental conditions than nuclear forces. The cyclic few-body atomic interactions also contribute to many-body dynamics. Braaten, Hammer, and Kusunoki discussed some effects in a Bose-Einstein condensate [6]. LeClair, Roman, and Sierra discussed a theoretical possibility of variation in the superconductivity mechanism [7]. On the other hand, the...
range of phenomena that may exhibit cycle structure extends also to sub-nuclear domain. Namely, Braaten and Hammer observed that masses of the up and down quarks appear to be close to special values at which an infrared cyclic behavior may develop in quantum chromodynamics [8]. From a mathematical point of view, it is also interesting that the cyclic behavior occurs in the elementary case of two-body quantum mechanics with the well-known potential $r^{-2}$, as recently described by Braaten and Phillips [9]. Readers interested in the literature that discusses quantum mechanics with short-range interactions without direct reference to the RG limit cycle can consult a review article by Nielsen, Fedorov, Jensen and Garrido [10]. Braaten and Hammer recently wrote a review article on universal cyclic properties of physical systems with two-body short range potentials and large scattering length in the context of renormalization of the potentials [11]. See also [12].

It should be mentioned that examples of renormalization group maps, which have critical attractors that are not simple fixed points, have been reported in the literature concerning classical dynamical systems. The author has learned about existence of a number of works during the process of preparation of this report [13–20].

The model Hamiltonian reviewed here has been discovered in our RG studies of Hamiltonians that were focused on hadronic parton dynamics in the infinite momentum frame [21]. We have recognized that the model has a RG cycle in a later article [22,23]. We have subsequently found an analytic solution for the RG behavior of Hamiltonians in the vicinity of the cycle in the model and this enabled us to discuss the RG universality of quantum Hamiltonians with a limit cycle using the model [24]. The purpose of my talk is to describe the cycle in our model, discuss corrections due to irrelevant operators, and show tuning to criticality, which allows the cycle spectrum to clearly appear even in approximations in which the basis of the space of quantum states is limited to about 70. This is attractive because one can inspect what happens in the model using a computer. A machine can provide numerical facts that help in developing intuition which otherwise is not available. I also address issues that are not fully understood and invite more research on mathematics of the RG limit cycle in quantum mechanics. Since atomic interactions can be strongly influenced by varying external fields (e.g., see the recent work of Roberts, Claussen, Cornish, and Wieman on magnetic field dependence of ultracold inelastic atomic collisions [25]), better understanding of quantum Hamiltonians near a RG limit cycle is a prerequisite to systematic experimental searches for cycles in real systems. The distinct feature to look for is the geometric sequence of bound states converging at threshold.