12 Degeneracy Algebras and Dynamical Algebras

12.1 Degeneracy Algebras

Another important application of algebraic methods in physics is to the study of exactly solvable problems in quantum mechanics. Consider quantum mechanics in \( \nu \) dimensions described by the Hamiltonian

\[
H = -\frac{\hbar^2}{2m} \nabla^2 + V(r)
\]  

(12.1)

where \( \nabla^2 \) is the Laplace operator and \( r \equiv (x_1, x_2, \ldots, x_\nu) \) denotes a vector in \( \nu \) dimensions with components \( x_1, x_2, \ldots, x_\nu \). This Hamiltonian is obtained from the classical Hamiltonian

\[
H = \frac{p^2}{2m} + V(r)
\]  

(12.2)

by the usual quantization procedure \( p \to \frac{\hbar}{i} \nabla \). If the Hamiltonian (12.1) can be written in terms of the Casimir operator \( C \) of an algebra \( g \),

\[
H = f(C)
\]  

(12.3)

the eigenvalue problem for \( H \) can be solved in explicit analytic form,

\[
E = \langle f(C) \rangle.
\]  

(12.4)

This situation is a dynamic symmetry, Sect. 11.2, except that only the Casimir operator of \( g \) and not those of the subalgebra chain \( g \supset g' \supset g'' \supset \ldots \) appears in (12.3). The representations \([\lambda]\) of \( g \) still label the eigenstates of the Hamiltonian and the symbol \( \langle \rangle \) denotes expectation value in the representation \([\lambda]\). If \( \text{dim}[\lambda] \neq 1 \), more than one state has energy \( E \). The state is said to be degenerate and the algebra \( g \) is called the \textit{degeneracy algebra}, \( g_c \), of the problem.

12.2 Degeneracy Algebras in \( \nu \geq 2 \) Dimensions

A particularly interesting class of problems is that of quantum mechanics in \( \nu \geq 2 \) dimensions with rotationally invariant potentials, \( V = V(r) \). Here

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This problem admits two and only two exactly solvable cases, the isotropic harmonic oscillator with \( V(r) = \frac{1}{2} kr^2 \), and the Coulomb (or Kepler) problem with \( V(r) = \frac{k}{r} \).

### 12.2.1 The Isotropic Harmonic Oscillator

The Hamiltonian operator, in units where \( \hbar = m = 1 \) and \( k = 1 \), is

\[
H = \frac{1}{2} (p^2 + r^2) = \frac{1}{2} (-\nabla^2 + r^2).
\]  

(12.6)

Introducing the bosonic realization of Chap. 7 written in differential form and generalizing the results of Example 3 one can write the linear Casimir operator of \( u(\nu) \) as

\[
C_1(u(\nu)) = \frac{1}{2} \sum_{j=1}^{\nu} \left( x_j - \frac{\partial}{\partial x_j} \right) \left( x_j + \frac{\partial}{\partial x_j} \right).
\]  

(12.7)

The basic commutation relations

\[
\left[ x_i, \frac{\partial}{\partial x_j} \right] = -\delta_{ij}
\]  

(12.8)

give

\[
H = C_1(u(\nu)) + \frac{\nu}{2}.
\]  

(12.9)

The degeneracy algebra of the \( \nu \)-dimensional harmonic oscillator is thus \( u(\nu) \). [J.M. Jauch and E.H. Hill, On the Problem of Degeneracy in Quantum Mechanics, Phys. Rev. 57, 641 (1940).] States are characterized by the totally symmetric irreducible representations \([n, 0, \ldots, 0] \equiv [n] \) of \( u(\nu) \), with eigenvalues

\[
E(n) = n + \frac{\nu}{2} \quad n = 0, 1, \ldots, \infty.
\]  

(12.10)

Although harmonic oscillator problems are best attacked by bosonic realizations of Lie algebras, it is still of interest to consider differential realizations in terms of coordinates \( r \equiv (x_1, x_2, \ldots, x_\nu) \) and momenta \( p = \frac{1}{i} \nabla \equiv \frac{1}{i} \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_\nu} \right) \).

**Example 1. Isotropic harmonic oscillator in three dimensions**

As discussed in Chap. 7, this problem is best solved in spherical coordinates \( r, \theta, \phi \). The Hamiltonian (12.2) is

\[
H = \frac{p^2}{2} + \frac{r^2}{2},
\]  

(12.11)

where \( r \) and \( p \) are here three-dimensional vectors. The nine operators