Wadge Degrees of $\omega$-Languages of Deterministic Turing Machines

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Abstract. We describe Wadge degrees of $\omega$-languages recognizable by deterministic Turing machines. In particular, it is shown that the ordinal corresponding to these degrees is $\xi^\omega$ where $\xi = \omega_1^{CK}$ is the first non-recursive ordinal known as the Church-Kleene ordinal. This answers a question raised in [Du0?]..

Keywords: Wadge degree, hierarchy, reducibility, $\omega$-language, Cantor space, set-theoretic operation.

1 Formulation of Main Result

In [Wag79] K. Wagner described a remarkable hierarchy of regular $\omega$-languages. In [Se94, Se95, Se98] the author gave a new exposition of the Wagner hierarchy relating it to the Wadge hierarchy and the fine hierarchy [Se95a]. In [Du0?] J. Duparc extended the Wagner hierarchy to deterministic context-free $\omega$-languages and formulated a conjecture about Wadge degrees of $\omega$-languages recognized by deterministic Turing machines.

Here we prove the conjecture of J. Duparc and show that Wadge degrees corresponding to deterministic acceptors form an easy structure. This is contrasted by a recent series of papers by O. Finkel showing that for the nondeterministic context-free $\omega$-languages the situation is more complicated.

Our methods come from computability theory and descriptive set theory. We assume some familiarity of the reader with such things as recursive hierarchies, countable and recursive ordinals etc.

Let $\{\Sigma_0^\alpha\}_{\alpha<\omega_1}$, where $\omega_1$ is the first uncountable ordinal, denote the Borel hierarchy of subsets of the Cantor space $2^\omega$ (all results below hold true also for the space $\{0,\ldots,n+1\}^\omega$ for any $n<\omega$ but for notational simplicity we consider only the case $n=0$) or the Baire space $\omega^\omega$. As usual, $\Pi_0^\omega$ denotes the dual class for $\Sigma_0^\omega$ while $\Delta_0^\omega = \Sigma_0^\omega \cap \Pi_0^\omega$ — the corresponding ambiguous class.

Let $B = \cup_{\alpha<\omega_1} \Sigma_0^\alpha$ denote the class of all Borel sets.

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In [Wad72, Wad84] W. Wadge described the finest possible topological classification of Borel sets by means of a relation \( \leq_W \) on subsets of a space \( S \in \{2^\omega, \omega^\omega\} \) defined by

\[ A \leq_W B \leftrightarrow A = f^{-1}(B), \]

for some continuous function \( f : S \to S \). He showed that the structure \((B; \leq_W)\) is well-founded, proved that for all \( A, B \in B \) either \( A \leq_W B \) or \( \bar{B} \leq_W A \) (we call structures satisfying these two properties, in which \( \bar{B} \) denotes the complement of a set \( B \), \textit{almost well-ordered}), and computed the corresponding (very large) ordinal \( \upsilon \). In [VW78, Ste81] it was shown that for any Borel set \( A \) which is non-self-dual (i.e., \( A \not\leq_W \bar{A} \)) exactly one of the principal ideals \( \{X | X \leq_W A\} \), \( \{X | X \leq_W \bar{A}\} \) has the separation property.

The results cited in the last paragraph give rise to the \textit{Wadge hierarchy of Borel sets} which is, by definition, the sequence \( \{\Sigma_\alpha\}_{\alpha<\upsilon} \) of all non-self-dual principal ideals of \((B; \leq_W)\) not having the separation property [Mo80] and satisfying for all \( \alpha < \beta < \upsilon \) the strict inclusion \( \Sigma_\alpha \subset \Delta_\beta \). As usual, we set \( \Pi_\alpha = \{X | X \in \Sigma_\alpha\} \) and \( \Delta_\alpha = \Sigma_\alpha \cap \Pi_\alpha \). Note that the classes \( \Sigma_\alpha \setminus \Pi_\alpha \), \( \Pi_\alpha \setminus \Sigma_\alpha \), \( \Delta_{\alpha+1} \setminus (\Sigma_\alpha \cup \Pi_\alpha) \) \((\alpha < \upsilon)\), which we call \textit{constituents} of the Wadge hierarchy, are exactly the equivalence classes induced by \( \leq_W \) on Borel subsets of the Cantor space. Please distinguish classes of the Wadge hierarchy (denoted without upper index) from corresponding classes of the Borel hierarchy (denoted with upper index 0).

There is a well-known small difference between the Wadge hierarchies in the Baire and in the Cantor space with respect to the question for which \( \alpha < \upsilon \) the class \( \Delta_\alpha \) has a \( W \)-complete set (such sets correspond to the self-dual Wadge degrees). For the Cantor space, these are exactly the successor ordinals \( \alpha < \upsilon \) while for the Baire space — the successor ordinals and the limit ordinals of countable cofinality [VW78]. This follows easily from the well-known fact that the Cantor space is compact while the Baire space is not.

The Wadge hierarchy on the Cantor space is of interest to the theory of \( \omega \)-languages since in this theory people also try to classify classes of \( \omega \)-languages according to their ‘complexity’. In [Se94, Se95, Se98] the Wagner hierarchy of regular \( \omega \)-languages was related to the Wadge hierarchy. The order type of Wadge degrees of regular \( \omega \)-languages is \( \omega^\omega \) [Wag79]. In [Du0?] the Wadge degrees of \( \omega \)-languages recognizable by deterministic push-down automata were determined; the corresponding ordinal is \( (\omega^\omega)^\omega \). In [Du0?] a conjecture on the structure of Wadge degrees of \( \omega \)-languages recognizable by deterministic Turing machines was formulated (for the Muller acceptance condition, see [Sta97]) implying that the corresponding ordinal is \( \xi^\omega \), where \( \xi = \omega_1^\omega \) is the first non-recursive ordinal known also as the Church-Kleene ordinal.

In this paper we prove the conjecture from [Du0?]. To formulate the corresponding result, define an increasing function \( e : \xi^\omega \to \omega_1^\omega \) by

\[ e(\xi^\omega \alpha_n + \cdots + \xi_1 \alpha_1 + \alpha_0) = \omega_1^\omega \alpha_n + \cdots + \omega_1^\omega \alpha_1 + \alpha_0, \]