4 Foundations of q-physics

We present the standard q-theory\(^1\) while, at each element, striving for the maximum likeness to Chap. 2 on Foundations of classical physics. We go slightly beyond the traditional treatment and, e.g., we define non-projective q-measurements as well as the phenomenon of entanglement. Leaf through Chap. 2 again, and compare!

4.1 State space, superposition

The state space of a q-system is a Hilbert space \( \mathcal{H} \), in case of \( d \)-state q-system it is the \( d \)-dimensional complex vector space:

\[
\mathcal{H} = \mathbb{C}^d = \{ c_\lambda; \lambda = 1, 2, \ldots, d \},
\]  

(4.1)

where the \( c_k \)'s are the elements of the complex column-vector in the given basis. The pure state of a q-system is described by a complex unit vector, also called state vector. In basis-independent abstract (Dirac-) notation it reads:

\[
|\psi\rangle \equiv \left[ \begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_d \end{array} \right], \quad \langle \psi | \equiv [ c_1^*, c_2^*, \ldots, c_d^* ], \quad \sum_{\lambda=1}^{d} |c_\lambda|^2 = 1.
\]  

(4.2)

The inner product of two vectors is denoted by \( \langle \psi | \varphi \rangle \). Matrices are denoted by a “hat” over the symbols, and their matrix elements are written as \( \langle \psi | \hat{A} | \varphi \rangle \). In q-theory, the components \( c_k \) of the complex vector are called probability amplitudes. Superposition, i.e. normalized complex linear combination of two or more vectors, yields again a possible pure state.

The generic state is mixed, described by trace-one positive semidefinite density matrix:

\[
\hat{\rho} = \{ \rho_{\lambda\mu}; \lambda, \mu = 1, 2, \ldots, d \} \geq 0, \quad \text{tr} \, \hat{\rho} = 1.
\]  

(4.3)

The generic state is interpreted on the statistical ensemble of identical systems. The density matrix of pure state (4.2) is a special case, it is the one-dimensional hermitian projector onto the subspace given by the state vector:

\(^1\) Cf. [6] by von Neumann.
\[ \hat{\rho}_{\text{pure}} = \hat{P} = |\psi\rangle \langle \psi| . \] (4.4)

We can see that multiplying the state vector by a complex phase factor yields the same density matrix, i.e., the same q-state. Hence the phase of the state vector can be deliberately altered, still the same pure q-state is obtained. In the conservative q-theory, contrary to the classical theory, not even the pure state is interpreted on a single system but on the statistical ensemble of identical systems.

### 4.2 Mixing, selection, operation

Random mixing the elements of two ensembles of q-states \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) at respective rates \( w_1 \geq 0 \) and \( w_2 \geq 0 \) yields the new ensemble of the q-state

\[ \hat{\rho} = w_1 \hat{\rho}_1 + w_2 \hat{\rho}_2; \quad 0 \leq w_1, w_2 \leq 1; \quad w_1 + w_2 = 1 . \] (4.5)

A generic mixed q-state can always be prepared (i.e. decomposed) as the mixture of two or more other mixed q-states in infinitely many different ways. After mixing, however, it is totally impossible to distinguish which way the mixed q-state was prepared. It is crucial, of course, that mixing must be probabilistic. A given mixed q-state can also be prepared (decomposed) as a mixture of pure q-states and this mixture is, contrary to the classical case, not unique in general.

Let operation \( M \) on a given q-state \( \hat{\rho} \) mean that we perform the same transformation on each q-system of the corresponding statistical ensemble. Mathematically, \( M \) is linear trace-preserving completely positive map, cf. Sect. 8.1, to bring an arbitrary state \( \hat{\rho} \) into a new state \( M \hat{\rho} \). Contrary to classical operations, not all positive maps correspond to realizable q-operations, but the completely positive ones. The operation’s categorical linearity follows from the linearity of the procedure of mixing (4.5). Obviously we must arrive at the same q-state if we mix two states first and then we subject the systems of the resulting ensemble to the operation \( M \) or, alternatively, we perform the operation prior to the mixing the two ensembles together:

\[ M (w_1 \hat{\rho}_1 + w_2 \hat{\rho}_2) = w_1 M \hat{\rho}_1 + w_2 M \hat{\rho}_2 . \] (4.6)

This is just the mathematical expression of the operation’s linearity.

Selection of a given ensemble into specific sub-ensembles, a contrary process of mixing, will be possible via so-called selective q-operations. They correspond mathematically to trace-reducing completely positive maps, cf. Sect. 8.3. The most typical selective q-operations are called q-measurements 4.4.

### 4.3 Equation of motion

Dynamical evolution\(^2\) of a closed q-system is determined by its hermitian Hamilton matrix \( \hat{H} \). The von Neumann equation of motion takes this form:

\(^2\)Our lectures use the Schrödinger-picture: the q-states \( \hat{\rho} \) evolve with \( t \), the q-physical quantities \( \hat{A} \) do not.