Deconvolution and Credible Intervals using Markov Chain Monte Carlo Method

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Abstract. In certain applications, e.g. during reconstruction of pulsatile hormone secretion, the traditional deterministic deconvolution techniques fail primarily due to ill conditioning. To overcome these problems, deconvolution was formulated using a stochastic approach within the Bayesian modelling framework. The stochastic deconvolution with a piece-wise constant definition of the signal (the input function) cannot be solved analytically but the solution was found by employing Markov chain Monte Carlo method. A computationally efficient sampling algorithm combined with a discrete deconvolution method was employed. An example analysis demonstrated the application of the stochastic deconvolution method to the estimation of hormone (insulin) secretion.

1 Introduction

Novel statistical computational approaches enable solutions to a wide range of problems, which cannot be handled by traditional methods. In particular, Markov chain Monte Carlo (MCMC) methods [1] have been applied to characterise multidimensional probability distributions in the areas of, for example, image analysis, population pharmacokinetics, and gene mapping [2, 3].

Deconvolution is a method for signal reconstruction and belongs to the family of model-independent approaches. Deconvolution estimates a large number of parameters as opposed to model-based methods, which use a model structure to reduce dimensionality of the parameter space.

Under certain conditions deconvolution becomes ill conditioned. The combination of frequent sampling and a slow (in relation to variations in the input function) unit impulse response causes the measurement error to be amplified. The calculated signal (the input function) then displays erratic behaviour and is sensitive to small variations in measurements.

Ill conditioning has been tackled by, for example, a regularisation method [4, 5]. Other methods have also been developed but all adopt additional assumption(s) about the underlying signal. Often, it is the assumption of smoothness of the signal. Smoothing carried out prior to processing (replacing measurements by moving averages) or during processing (introduction of a regularisation component) avoids successfully ill conditioning but also looses information about variability of the signal.
All information could be retained if deconvolution is formulated in probabilistic terms. Deconvolution then calculates the signal but also propagates the measurement error through the deconvolution operation to the signal estimate. This is the objective of stochastic deconvolution.

The present work describes the formulation of stochastic deconvolution and uses an MCMC method to calculate the solution in an example taken from a biomedical field.

2 Methodology

2.1 Discrete Convolution

The general convolution integral has the form

\[ c(t) = \int_{-\infty}^{t} u(t - \tau) r(\tau) d\tau + \epsilon(t) \]  \tag{1}

where \( c(t) \) represents the measurement, \( u(t) \) represents the unit impulse response, \( r(t) \) represents the input function, and \( \epsilon(t) \) represents the measurement error. It will be assumed further in the text that the unit response is described by a sum of exponentials, i.e. \( u(t) = \sum_{i=1}^{K} A_i e^{-a_i t} \), as traditional in the biomedical field.

The discrete version of the convolution integral treats the input as a piecewise constant function, which is specified by a time series \( r = \{r_i\}, i = 1 \ldots N \).

The discrete version of the convolution integral given by Eq. 1 is written as

\[ c = Ar + \epsilon, \]  \tag{2}

where \( c = \{c_j\}, j = 1 \ldots M \), is a set of measurements, \( \epsilon = \{\epsilon_j\}, j = 1 \ldots M \), is a set of measurement errors, and \( A \) is an \( M \times N \) convolution matrix [6]. The measurement errors \( \epsilon_j \) are assumed to be independent and normally distributed with a constant coefficient of variation CV, \( \epsilon_j \sim N(0, \frac{\text{CV}}{100} c_j^2) \).

Elements \( a_{ij} \) of \( A \) are calculated as \( a_{ij} = \int_{t_{i-1}}^{t_i} u(t_i - \tau) d\tau \) assuming that the measurement time grid and the input function time grid coincide (i.e. \( N = M \)).

2.2 Stochastic Deconvolution

The stochastic formulation of convolution gives stochastic interpretation to the input function \( r \). The Bayesian modelling framework was employed in the formulation.

In the present formulation, components \( r_i \) of the input function are treated as random variables, which are assumed to be identically distributed with a non-informative, uniform prior distribution

\[ p(r_i) = \begin{cases} \text{const} & \text{if } r_{\text{max}} \geq r_i \geq 0 \\ 0 & \text{otherwise,} \end{cases} \]