Compositional Characterizations of \( \lambda \)-Terms Using Intersection Types *
(Extended Abstract)

M. Dezani-Ciancaglini\textsuperscript{1}, F. Honsell\textsuperscript{2}, and Y. Motohama\textsuperscript{1}

\textsuperscript{1} Dipartimento di Informatica, Università di Torino
Corso Svizzera 185, 10149 Torino, Italy
\texttt{\{dezani,yoko\}@di.unito.it}

\textsuperscript{2} Dipartimento di Matematica ed Informatica, Università di Udine
Via delle Scienze 208, 33100 Udine, Italy
\texttt{honsell@dimi.uniud.it}

Abstract. We show how to characterize compositionally a number of evaluation properties of \( \lambda \)-terms using Intersection Type assignment systems. In particular, we focus on termination properties, such as strong normalization, normalization, head normalization, and weak head normalization. We consider also the persistent versions of such notions. By way of example, we consider also another evaluation property, unrelated to termination, namely reducibility to a closed term.

Many of these characterization results are new, to our knowledge, or else they streamline, strengthen, or generalize earlier results in the literature. The completeness parts of the characterizations are proved uniformly for all the properties, using a set-theoretical semantics of intersection types over suitable kinds of stable sets. This technique generalizes Krivine’s and Mitchell’s methods for strong normalization to other evaluation properties.

Introduction

The intersection-types discipline was introduced in \cite{Dezani-Ciancaglini1999} as a tool of overcoming the limitations of Curry’s type assignment system. Subsequently it was used in \cite{Dezani-Ciancaglini2000} as a tool for proving Scott’s conjecture concerning the completeness of the set-theoretic semantics for simple types.

Very early on, however, it was realized that intersection type theories are a very expressive tool for giving compositional characterizations (i.e. characterizations based on properties of proper subterms) of evaluation properties of \( \lambda \)-terms. There are two seminal results in this respect.

The first result is that the \( \Omega \)-free fragment of intersection-types allows one to type all and only the strongly normalizing terms. This is largely a folklore result; the first published proof appears in \cite{Dezani-Ciancaglini2001}.

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The second result is the filter model construction based on the intersection type theory $\Sigma^{Bc\alpha}$, carried out in [6]. This result shows that there is a very tight connection between intersection types and compact elements in $\omega$-algebraic denotational models of $\lambda$-calculus. This connection later received a categorically principled explanation by Abramsky in the broader perspective of “domain theory in logical form” [1].

Since then, the number of intersection type theories, used for the fine study of the denotational semantics of untyped $\lambda$-calculus, has increased considerably (e.g. [11,10,16,13,12,20,15]). In all these cases the corresponding intersection type assignment systems are used to provide finite logical presentations of particular domain models, which can thereby be viewed also as filter models. And hence, intersection type theories provide characterizations of particular semantical properties.

In this paper we address the problem of investigating uniformly the use of intersection type theories, and corresponding type assignment systems, for giving a compositional characterization of evaluation properties of $\lambda$-terms.

In particular we discuss termination properties such as strong normalization, normalization, head normalization, weak head normalization. We consider also the persistent versions of such notions (see Definition 8). By way of example we consider also another evaluation property, unrelated to termination, namely reducibility to a closed term.

Many of the characterization results that we give are indeed inspired by earlier semantical work on filter models of the untyped $\lambda$-calculus, but they are rather novel in spirit. We focus, in fact, on proof-theoretic properties of intersection type assignment systems per se. Most of our characterizations are therefore new, to our knowledge, or else they streamline, strengthen, or generalize earlier results in the literature.

The completeness part of the characterizations is proved uniformly for all the properties. We use a very elementary presentation of the technique of logical relations phrased in terms of a set-theoretical semantics of intersection types over suitable kinds of stable sets. This technique generalizes Krivine’s [17] and Mitchell’s [19] proof methods for strong normalization, to other evaluation properties.

The paper is organized as follows. In Section 1 we introduce the intersection type language, intersection type theories and type assignment systems. In Section 2 we introduce the various properties of $\lambda$-terms we shall focus on. In Section 3 we give the compositional characterizations of such properties. Final remarks and open problems appear in Section 4.

1 Intersection Type Theories and Type Assignment Systems

Intersection types are syntactical objects which are built inductively by closing a given set $C$ of type atoms (constants) under the function type constructor $\to$ and the intersection type constructor $\cap$. 