Deciding Clique-Width for Graphs of Bounded Tree-Width
(Extended Abstract)

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Abstract. We show that there exists a linear time algorithm for deciding whether a graph of bounded tree-width has clique-width \( k \) for some fixed integer \( k \).

1 Introduction

The clique-width of a graph is defined by a composition mechanism for vertex-labeled graphs, see [CO00]. The operations are the vertex disjoint union, the addition of edges between all pairs of vertices with a given pair of labels, and the relabeling of vertices. The clique-width of a graph \( G \) is the minimum number of labels needed to define \( G \). Graphs of bounded clique-width are especially interesting from an algorithmic point of view. A lot of NP-complete graph problems can be solved in polynomial time for graphs of bounded clique-width if the composition of the graph is explicitly given. For example, all graph properties which are expressible in monadic second order logic with quantifications over vertices and vertex sets (MSO\(_1\)-logic) are decidable in linear time on graphs of bounded clique-width, see [CMR00]. The MSO\(_1\)-logic has been extended by counting mechanisms which allow the expressibility of optimization problems concerning maximal or minimal vertex sets, see [CMR00]. All these graph problems expressible in extended MSO\(_1\)-logic can be solved in polynomial time on graphs of bounded clique-width. Furthermore, a lot of NP-complete graph problems which are not expressible in MSO\(_1\)-logic or extended MSO\(_1\)-logic like Hamiltonicity and certain partitioning problems can also be solved in polynomial time on graphs of bounded clique-width, see [KR01,Wan94].

If a graph \( G \) has clique-width at most \( k \) then the edge complement \( \overline{G} \) has clique-width at most \( 2k \), see [CO00]. Distance hereditary graphs have clique-width at most 3, see [GR00]. The set of all graphs of clique-width at most 2 is the set of all labeled cographs. The clique-width of permutation graphs, interval graphs, grids and planar graphs is not bounded by some fixed integer \( k \), see [GR00]. An arbitrary graph with \( n \) vertices has clique-width at most \( n - r \), if \( 2^r < n - r \), see [Joh98]. The recognition problem for graphs of clique-width

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at most $k$ is still open for $k \geq 4$. Clique-width of at most 3 is decidable in polynomial time, see [CHL+00]. Clique-width of at most 2 is decidable in linear time, see [CPS85].

A famous class of graphs for which a lot of NP-complete graph problems can be solved in polynomial time is the class of graphs of bounded tree-width, see Bodlaender [Bod98] for a survey. For every fixed integer $k$, it is decidable in linear time whether a given graph $G$ has tree-width $k$, see [Bod96]. All graph properties expressible in monadic second order logic with quantifications over vertex sets and edge sets (MSO$_2$-logic) are decidable in linear time for graphs of bounded tree-width by dynamic programming, see [Cou90]. The MSO$_2$-logic has also been extended by counting mechanisms to express optimization problems which can then be solved in polynomial time for graphs of bounded tree-width, see [ALS91].

Every graph of tree-width at most $k$ has clique-width at most $2^{k+1} + 1$, see [CO00]. Since the set of all cographs already contains all complete graphs, the set of all graphs of clique-width at most 2 does not have bounded tree-width. In [GW00], it is shown that every graph of clique-width $k$ which does not contain the complete bipartite graph $K_{n,n}$ for some $n > 1$ as a subgraph has tree-width at most $3k(n-1) - 1$.

A simple algorithm to decide a graph property on a graph of bounded tree-width can be obtained from a partition of all $l$-terminal graphs into a finite number of equivalence classes, see, for example, [Bod98]. An $l$-terminal graph is a graph with a list of $l$ distinct vertices called terminals. Two $l$-terminal graphs $G$ and $H$ can be combined to a graph $G \circ H$ by taking the disjoint union of $G$ and $H$ and then identifying the $i$-th terminal of $G$ with the $i$-th terminal of $H$ for $1 \leq i \leq l$. They are called equivalent with respect to a graph property $\Pi$ if for all $l$-terminal graphs $J$ the answer to $\Pi$ is the same for $G \circ J$ and $H \circ J$. A graph property $\Pi$ is decidable in linear time on a graph of bounded tree-width if there is a finite number of equivalence classes with respect to $\Pi$ for all $l$-terminal graphs and all $l \geq 0$. The linear time algorithm first computes a binary tree-decomposition $T$ for $G$ and then bottom-up the equivalence class for every $l$-terminal graph $G'$ represented by a complete subtree $T'$ of $T$. The equivalence class of $G'$ defined by subtree $T'$ with root $u'$ is computable in time $O(1)$ from the classes of the two $l$-terminal graphs defined by the two subtrees in $T'-\{u'\}$.

In this paper, we prove that the graph property “clique-width at most $k$” divides the set of all $l$-terminal graphs into a finite number of equivalence classes. This implies that there exists a linear time algorithm for deciding “clique-width at most $k$” for graphs of bounded tree-width. Since every graph of tree-width $r$ has clique-width at most $2^{r+1} + 1$, there is also a linear time algorithm for computing the clique-width of a graph of bounded tree-width by testing “clique-width at most $k$” for $k = 1, \ldots, 2^{r+1} + 1$. The proofs of lemma 1, 5, 6, 7, and theorem 1 are omitted due to space limitations. Note that it remains open whether the clique-width $k$ property is expressible in MSO$_2$-logic and whether “clique-width at most $k$” is decidable in polynomial time for arbitrary graphs.

\footnote{A complete version can be found at www.cs.uni-duesseldorf.de/~wanke.}