Geodesic Interpolating Splines

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Abstract. We propose a simple and efficient method to interpolate landmark matching by a non-ambiguous mapping (a diffeomorphism). This method is based on spline interpolation, and on recent techniques developed for the estimation of flows of diffeomorphisms. Experimental results show interpolations of remarkable quality. Moreover, the method provides a Riemannian distance on sets of landmarks (with fixed cardinality), which can be defined intrinsically, without referring to diffeomorphisms. The numerical implementation is simple and efficient, based on an energy minimization by gradient descent. This opens important perspectives for shape analysis, applications in medical imaging, or computer graphics.

1 Introduction

This paper proposes a new, efficient and consistent method for generating dense diffeomorphisms within an image from sparse information on the displacements of a finite number of points (landmarks). This is an important issue for image processing and computer graphics, and the problem has generated a large number of publications, starting with the seminal papers of Bookstein (see [3] and references therein). There are numerous applications: generating deformations from the position of control points is used, for example, to synthesize facial expressions, or to compute morphings; analyzing variations of shape has application in medical imaging or face recognition, matching is essential for the construction of anatomical atlases. Jointly with the purpose of interpolating from landmark-matching, comes the issue of measuring the discrepancy between two groups of matched landmarks. This is not an obvious problem, and it seems quite intuitive that the smoothness of the underlying, unobserved, global displacement comes as an essential part for the perceptive impression of discrepancy. A third, important, feature is the consistency of the interpolated displacement, in the sense that it should be one-to-one, ensuring that there cannot be two distinct parts of the original picture which are matched to the same zone in the target.

The method which is described here addresses the three problems simultaneously. It does provide a way to interpolate from landmark-matching, while providing a distance between configurations of landmarks which takes into account the smoothness of the underlying warping, generated as a diffeomorphism.
defined on the image grid. As a fourth, non-negligible property, comes the fact that this method is easy to implement, and numerically efficient.

To fix notations, let $\Omega$ be a bounded set in the plane. When $(x_1, \ldots, x_N), (y_1, \ldots, y_N)$, two sets of $N$ labeled landmarks in $\Omega$, are given, we shall deal with the problem of finding a diffeomorphism $g$ of $\Omega$, with minimal size (in a sense to be defined), such that, for all $i$, $g(x_i) \simeq y_i$ (inexact matching).

The method which is developed in the sequel takes its roots from three main ideas:

- Interpolating splines, as pioneered by Bookstein in computer vision ([3]), and widely used to generate dense warping from sparse information.
- Generation of diffeomorphisms as flows (solutions of an ODE), in a framework which guarantees smoothness and consistency, as in [10, 4].
- Computation of geodesic distances (minimal path length) on deformable data, as used in [11, 8].

The analysis results in a simple algorithm to compute diffeomorphisms from landmark data.

The paper is organized as follows. We start by reviewing the elements of spline theory which will be needed, and relate them to the (non-diffeomorphic) interpolation introduced by Bookstein. This forms the first ingredient for our method. In a second step, we give a presentation of the theory of groups of diffeomorphisms, generated as flows (solutions of ODEs) on a set $\Omega$, and show how this framework can be used to generate geodesic distances on structures acted on by diffeomorphisms (ie. deformable patterns). The last step will be to use this framework on the very simple deformable structure which are sets of landmarks, to derive an efficient algorithm for simultaneously computing distances between sets of landmarks and generating a diffeomorphism to interpolate the pointwise matching. The paper ends with a presentation of some experimental facts and data.

2 Landmark matching and splines

2.1 Splines

Like for all landmark-matching methods, the numerical efficiency of the algorithm that we propose relies on spline interpolation theory. For completeness of the presentation, we spend some time in describing the foundations of this theory, exhibiting in particular its remarkable algebraic simplicity.

Formally speaking, spline fitting can be considered a particular case of what follows. Let $\mathcal{H}$ be a Hilbert space, let $f_1, \ldots, f_N \in \mathcal{H}$, and $c_1, \ldots, c_N \in \mathbb{R}$ be given. Denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathcal{H}$. Consider the following problems:

1. Find $h \in \mathcal{H}$ such that $\|h\|$ is minimum subject to the constraints $\langle f_i, h \rangle = c_i$ for $i = 1, \ldots, N$. 
