Fully Complete Minimal PER Models
for the Simply Typed λ-Calculus

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Abstract. We show how to build a fully complete model for the maximal theory of the simply typed λ-calculus with $k$ ground constants, λ\textsubscript{k}. This is obtained by linear realizability over an affine combinatory algebra of partial involutions from natural numbers into natural numbers. For simplicity, we give the details of the construction of a fully complete model for λ\textsubscript{k} extended with ground permutations. The fully complete minimal model for λ\textsubscript{k} can be obtained by carrying out the previous construction over a suitable subalgebra of partial involutions. The full completeness result is then put to use in order to prove some simple results on the maximal theory.

Introduction

A categorical model of a type theory (or logic) is said to be fully-complete ([AJ94a]) if, for all types (formulae) $A, B$, all morphisms $f : [A] \rightarrow [B]$, from the interpretation of $A$ into the interpretation of $B$, are denotations of a proof-term of the entailment $A \vdash B$, i.e. if the interpretation function from the category of syntactical objects to the category of denotations is full. The notion of full-completeness is a counterpart to the notion of full abstraction for programming languages. A fully complete model indicates that there is a very tight connection between syntax and semantics. Equivalently, one can say that the term model has been made into a mathematically respectable structure.

Over the past decade, Game Semantics has been used successfully by various people to define fully-complete models for various fragments of Linear Logic, and to give fully-abstract models for many programming languages, including PCF, and other functional and non-functional languages. Recently, a new technique, called linear realizability (see [AL99,AL00]), has been proposed as a valid and less complex alternative to Game Semantics in providing fully complete and fully abstract models. In particular, this technique has been used in [AL99,AL00] to define a model fully complete w.r.t. the fragment of system F consisting of

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ML polymorphic types, and in [AL99a] to provide a fully complete model for
PCF. The linear (linear affine) realizability technique amounts to constructing
a category of Partial Equivalence Relations (PERs) over a Linear Combinatory
Algebra, LCA, (Affine Combinatory Algebra, ACA). This category turns out to
be linear (affine), and to form an adjoint model with its co-Kleisli category. The
notion of Linear (Affine) Combinatory Algebra introduced by the first author
([Abr97]) refines the standard notion of Combinatory Algebra, in the same way
in which intuitionistic linear (affine) logic refines intuitionistic logic. The con-
struction of PER models from LCA’s (ACA’s) of [AL99,AL00] is quite simple
and clear, and it yields models with extensionality properties. Many examples
of linear combinatory algebras arise in the context of Abramsky’s categorical
version of Girard’s Geometry of Interaction ([AJ94,Abr97,Abr96,AHS98]).

In this paper, we define a fully complete PER model for the maximal theory
≈ on $\lambda_k$. $\lambda_k$ is the simply typed $\lambda$-calculus with finitely many ground constants
in the ground type $\alpha$. The theory ≈ equates two closed $\lambda$-terms $M, N$ of type
$T_1 \to \ldots \to T_n \to \alpha$ if and only if, for all $P_1 \approx Q_1$ of type $T_1, \ldots, P_n \approx Q_n$ of
type $T_n$, $MP_1 \ldots P_n = \beta NQ_1 \ldots Q_n$. To our knowledge, our model is the first
model of ≈ different from the term model.

For simplicity, we show first how to build a fully complete minimal PER
model $M_{\alpha_k}$ for the simply typed $\lambda$-calculus with $k$ ground constants extended
with permutations of ground type. The fully complete minimal model for $\lambda_k$ can
then be obtained by cutting down the combinatory algebra. The model $M_{\alpha_k}$
for the extended language arises from the special affine combinatory algebra of
partial involutions used in [AL99,AL00] for modeling System F. It consists, es-
sentially, of the hierarchy of simple PERs over a PER having exactly $k$ distinct
equivalence classes (for any $k \geq 2$). The proof of full completeness carried out
in this paper is based on the linear affine analysis of the intuitionistic arrow,
which is possible in our PER category. Our proof uses a Decomposition The-
orem, which is by now a standard tool in discussing full completeness. In the
present case, given a partial involution which inhabits a PER interpreting a sim-
ple type, the Decomposition Theorem allows to recover the top-level structure
(up-to permutations) of the (possibly infinite) typed Böhm tree corresponding
to the given partial involution. Once we have the Decomposition Theorem, in
order to prove $\lambda$-definability, we still need to rule out possibly infinite typed
Böhm trees from the model. In order to do this, we prove an Approximation Theorem,
and we study an intermediate PER model $M_{\alpha_k}$ for $\lambda_k$ extended with
a new ground constant $\bot$, intended to denote the undefined constant. A variant
of this model, $M_{\alpha}$, for the special case of two ground constants, $\top$ and $\bot$, has
been used implicitly as an intermediate construction in the work of [AL99] on
system F.

In order to get a fully complete minimal model for $\lambda_k$, we only need to cut
down the algebra of involutions by putting an extra constraint, which allows us
to rule out permutations from the model $M_{\alpha_k}$.

The full completeness result is then put to use to prove (or re-prove) some
simple facts on the maximal $\lambda$-theory. In particular, the Context Lemma follows