On Algebraic Expressions of Series-Parallel and Fibonacci Graphs

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Abstract. The paper investigates relationship between algebraic expressions and graphs. Throughout the paper we consider two kinds of digraphs: series-parallel graphs and Fibonacci graphs (which give a generic example of non-series-parallel graphs). Motivated by the fact that the most compact expressions of series-parallel graphs are read-once formulae, and, thus, of $O(n)$ length, we propose an algorithm generating expressions of $O(n^2)$ length for Fibonacci graphs. A serious effort was made to prove that this algorithm yields expressions with a minimum number of terms. Using an interpretation of a shortest path algorithm as an algebraic expression, a symbolic approach to the shortest path problem is proposed.

1 Introduction

A graph $G = (V, E)$ consists of a vertex set $V$ and an edge set $E$, where each edge corresponds to a pair $(v, w)$ of vertices. If the edges are ordered pairs of vertices (i.e., the pair $(v, w)$ is different from the pair $(w, v)$), then we call the graph directed or digraph; otherwise, we call it undirected. A path from vertex $v_0$ to vertex $v_k$ in a graph $G = (V, E)$ is a sequence of its vertices $v_0, v_1, v_2, \ldots, v_{k-1}, v_k$, such that $(v_{i-1}, v_i) \in E$ for $1 \leq i \leq k$. A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A graph $G$ is homeomorphic to a graph $G'$ (homeomorph of $G'$) if $G$ can be obtained by subdividing edges of $G'$ via adding new vertices. A two-terminal directed acyclic graph (st-dag) has only one source $s$ and only one sink $t$. In an st-dag, every vertex lies on some path from the source to the sink.

An algebraic expression is called an st-dag expression if it is algebraically equivalent to the sum of products corresponding to all possible paths between the source and the sink of the st-dag \footnote{C.S. Calude et al. (Eds.): DMTCS 2003, LNCS 2731, pp. 215–224, 2003. © Springer-Verlag Berlin Heidelberg 2003}. This expression consists of terms (edge labels) and the operators $+$ (disjoint union) and $\cdot$ (concatenation, also denoted by juxtaposition when no ambiguity arises).

We define the total number of terms in an algebraic expression, including all their appearances, as the complexity of the algebraic expression. The complexity of an st-dag expression is denoted by $T(n)$, where $n$ is the number of vertices in
the graph (the size of the graph). An expression of the minimum complexity is called an optimal representation of the algebraic expression.

A series-parallel (SP) graph is defined recursively as follows:

(i) A single edge \((u, v)\) is a series-parallel graph with source \(u\) and sink \(v\).
(ii) If \(G_1\) and \(G_2\) are series-parallel graphs, so is the graph obtained by either of the following operations:

(a) Parallel composition: identify the source of \(G_1\) with the source of \(G_2\) and the sink of \(G_1\) with the sink of \(G_2\).

(b) Series composition: identify the sink of \(G_1\) with the source of \(G_2\).

The construction of a series-parallel graph in accordance with its recursive definition may be represented by a binary tree which is called a decomposition tree. The edges of the graph are represented by the leaves of the tree. The inner nodes of the tree are labeled \(S\), indicating a series composition, or \(P\) indicating a parallel composition. Each subtree in the decomposition tree corresponds to a series-parallel subgraph. Figure 1 shows an example of a series-parallel graph (a) together with its decomposition tree (b).