Parallel Performance of a 3D Elliptic Solver

Ivan Lirkov¹, Svetozar Margenov¹, and Marcin Paprzycki²

¹ Central Laboratory for Parallel Processing, Bulgarian Academy of Sciences
Acad.G.Bonchev, block 25A, 1113 Sofia, Bulgaria
{ivan,margenov}@cantor.bas.bg

² Department of Computer Science and Statistics, University of Southern Mississippi,
Hattiesburg, Mississippi, 39406-5106, USA
marcin@orca.st.usm.edu

Abstract. It was recently shown that block-circulant preconditioners applied to a conjugate gradient method used to solve structured sparse linear systems arising from 2D or 3D elliptic problems have good numerical properties and a potential for high parallel efficiency. In this note parallel performance of a circulant block-factorization based preconditioner applied to a 3D model problem is investigated. The aim of the presentation is to report on the experimental data obtained on SUN Enterprise 3000, SGI/Cray Origin 2000, Cray J-9x, Cray T3E computers and on two PC clusters.

1 Introduction

Let us consider numerical solution of a self-adjoint second order 3D linear boundary value problem of elliptic type. After discretization, such a problem results in a linear system $Ax = b$, where $A$ is a sparse symmetric positive definite matrix. In the computational practice, large-scale problems of this class are most often solved by Krylov subspace iterative (e.g. conjugate gradient) methods. Each step of such a method requires only a single matrix-vector product and allows exploitation of sparsity of $A$. The rate of convergence of these methods depends on the condition number $\kappa$ of the matrix $A$ (smaller $\kappa(A)$ results in faster convergence). Unfortunately, for second order 3D elliptic problems, usually $\kappa(A) = O(N^{2/3})$, where $N$ is the size of the discrete problem, and hence it grows rapidly with $N$. To alleviate this problem, iterative methods are almost always used with a preconditioner $M$. The preconditioner is chosen with two criteria in mind: to minimize $\kappa(M^{-1}A)$ and to allow efficient computation of the product $M^{-1}v$ for any given vector $v$. These two goals are often in conflict and a lot of research has been done devising preconditioners that strike a balance between them. Recently, a third aspect has been added to the above two, namely, the parallel efficiency of the iterative method (and thus the preconditioner).

One of the most popular and the most successful preconditioners are the incomplete LU (ILU) factorizations. Unfortunately, standard ILU preconditioners have limited degree of parallelism. Some attempts to modify them and introduce more parallelism often result in a deterioration of the convergence rate. R. Chan

© Springer-Verlag Berlin Heidelberg 2001
and T. F. Chan [2] proposed another class of preconditioners based on averaging coefficients of $A$ to form a block-circulant approximation. The block-circulant preconditioners are highly parallelizable but they are very sensitive to a possible high variation of the coefficients of the elliptic operator. To reduce this sensitivity a new class of circulant block-factorization (CBF) preconditioners [5] was introduced by Lirkov, Margenov and Vassilevski. Recently a new CBF preconditioner for 3D problems was introduced in [3,4].

The main goal of this note is to report on the parallel performance of the PCG method with a circulant block-factorization preconditioner applied to a model 3D linear PDE of elliptic type. Results of experiments performed on Sun Ultra-Enterprise, Crays J-9x and T3E, SGI/Cray Origin 2000 high performance computers and on two PC clusters are presented and analyzed.

We proceed as follows. In Section 2 we sketch the algorithm of the parallel preconditioner (for more details see [3,4]). Section 3 contains the theoretical estimate of its arithmetical complexity. Finally, in Section 4 we report the results of our experiments.

2 Circulant Block-Factorization

Let us recall that a circulant matrix $C$ has the form $(C_{k,j}) = (c_{(j-k) \mod m})$, where $m$ is the dimension of $C$. Let us also denote by $C = (c_0, c_1, \ldots, c_{m-1})$ the circulant matrix with the first row $(c_0, c_1, \ldots, c_{m-1})$. Any circulant matrix can be factorized as $C = FA F^*$ where $A$ is a diagonal matrix containing the eigenvalues of $C$, and $F$ is the Fourier matrix of the form

$$F_{jk} = \frac{1}{\sqrt{m}} e^{2\pi i \frac{jk}{m}},$$

where $F^* = F^T$ denotes the adjoint matrix of $F$.

The CBF preconditioning technique incorporates the circulant approximations into the framework of LU block-factorization. Let us consider a 3D elliptic problem (see also [3]) on the unit cube with Dirichlet boundary conditions. If the domain is discretized on a uniform grid with $n_1$, $n_2$ and $n_3$ grid points along the coordinate directions, and if a standard (for such a problem) seven-point FDM (FEM) approximation is used, then the stiffness matrix $A$ admits a block-tridiagonal structure. The matrix $A$ can be written in the form

$$A = \text{tridiag}(-A_{i,i-1}, A_{i,i}, -A_{i,i+1}) \quad i = 1, 2, \ldots, n_1,$$

where $A_{i,j}$ are block-tridiagonal matrices which correspond to the $x_1$-plane and the off-diagonal blocks are diagonal matrices. In this case the general CBF preconditioning approach is applied to construct the preconditioner $M_{CBF}$ in the form

$$M_{CBF} = \text{tridiag}(-C_{i,i-1}, C_{i,i}, -C_{i,i+1}) \quad i = 1, 2, \ldots, n_1,$$

where $C_{i,j} = \text{Block-Circulant}(A_{i,j})$ is a block-circulant approximation of the corresponding block $A_{i,j}$. The stiffness matrix $A$ and the preconditioner $M_{CBF}$