A Finitary Subsystem of the Polymorphic
\(\lambda\)-Calculus

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Abstract. We give a finitary normalisation proof for the restriction of system \(F\) where we quantify only over first-order type. As an application, the functions representable in this fragment are exactly the ones provably total in Peano Arithmetic. This is inspired by the reduction of \(\Pi_1\)-comprehension to inductive definitions presented in [Buch2] and this complements a result of [Leiv]. The argument uses a finitary model of a fragment of the system \(AF_2\) considered in [Kriv,Leiv].

1 The Polymorphic \(\lambda\)-Calculus

We let \(D\) be the set of all untyped, maybe open, \(\lambda\)-terms, with \(\beta\)-conversion as equality. We let \(c_n\) be the lambda term \(\lambda x \lambda f \, f^n \, x\). We consider the following types

\[ T ::= \alpha | T \to T | (\Pi \alpha)T \]

where in the quantification, \(T\) has to be built using only \(\alpha\) and \(\to\). We use \(T, U, V\) to denote over types.

We use the notation \(T_1 \to T_2 \to T_3\) for \(T_1 \to (T_2 \to T_3)\) and similarly \(T_1 \to T_2 \to \ldots \to T_n\) for \(T_1 \to (T_2 \to (\ldots \to T_n))\).

Let us give some examples to illustrate the restriction on quantification. We can have \(T = (\Pi \alpha)[\alpha \to \alpha]\) or \((\Pi \alpha)[\alpha \to (\alpha \to \alpha) \to \alpha]\) or even \((\Pi \alpha)[((\alpha \to \alpha) \to \alpha) \to \alpha]\) but a type such as \((\Pi \alpha)[(\Pi \beta)[\alpha \to \beta]] \to \alpha]\) is not allowed.

We have the following typing rules

\[
\frac{x : T \in \Gamma}{\Gamma \vdash x : T} \quad \frac{x : T \in \Gamma}{\Gamma, x : T \vdash t : U} \quad \frac{\Gamma \vdash u : V \to T}{\Gamma \vdash u : V} \quad \frac{\Gamma \vdash v : V}{\Gamma \vdash v : V} \quad \frac{\Gamma \vdash \lambda x \ t : T \to U}{\Gamma \vdash \lambda x \ t : T \to U} \quad \frac{\Gamma \vdash t : (\Pi \alpha)T}{\Gamma \vdash t : (\Pi \alpha)T} \quad \frac{\Gamma \vdash t : T}{\Gamma \vdash t : T(U)} \quad \frac{\Gamma \vdash u \, v : T \to T}{\Gamma \vdash u \, v : T}
\]

where \(\Gamma\) is a type context, i.e. an assignment of types to a finite set of variables, and in the last rule, \(\alpha\) does not appear free in any type of \(\Gamma\). We write \(T(U)\) for
a substitution where the variable which is substituted for is obvious form the context.

We let \( N \) be the type \( (\Pi \alpha)[\alpha \to (\alpha \to \alpha) \to \alpha] \). We have \( \vdash c_n : N \) for each \( n \). The goal of this note is to provide a finitary proof of the following result.

**Theorem 1.** If \( \vdash t : N \to N \) then for each \( n \) there exists \( m \) such that \( t \ c_n \ x \ f = f^m \ x \) for \( x, f \) variables.

This result can be seen as a special case of the normalisation property. We concentrate on this simplified case to illustrate the principle of our argument. From a proof theoretical view point this special case is as hard as the normalisation property.

This result follows from [Leiv] if in the formation of \( (\Pi \alpha)T \) we restrict \( T \) to be of rank \( \leq 2 \). We extend this to cover types such as

\[
(\Pi \alpha)[((\alpha \to \alpha) \to \alpha) \to \alpha]
\]

One non finitary proof of this result is the following. Each type is interpreted by a subset of \( D \). We define \( [T]_{\rho} \subset D \) where \( \rho \) is a function assigning subsets of \( D \) to type variables.

\[
[T \to U]_{\rho} = \{v \in D \mid (\forall t \in [T]_{\rho}) \ v \ t \in [U]_{\rho}\}
\]

and

\[
[(\Pi \alpha)T] = \bigcap_{X \subseteq D} [T]_{\rho, \alpha=X}
\]

We prove then, by induction on derivations

**Lemma 1.** If \( x_1 : T_1, \ldots, x_n : T_n \vdash t : T \) and \( u_i \in [T_i] \) then \( t(u_1, \ldots, u_n) \in [T] \)

**Corollary 1.** If \( \vdash t : N \) then \( t \in [N] \)

**Lemma 2.** If \( u \in [N] \) then there exists \( m \) such that \( u \ x \ f = f^m \ x \) for \( x, f \) variables.

**Proof.** Consider the subset \( S = \{t \mid (\exists m) \ t = f^m \ x\} \). We have \( x \in S \) and \( f \ t \in S \) if \( t \in S \). Hence the result.

We can now prove the theorem. If \( \vdash t : N \to N \) we have then \( \vdash t \ c_n : N \) because \( \vdash c_n : N \). But this implies, by the two lemmas that there exist \( m \) such that \( t \ c_n \ x \ f = f^m \ x \). Let us write \( m = \phi(n) \). We say then that the function \( \phi \) is represented by the term \( t \).