The Girard-Reynolds Isomorphism

Philip Wadler
Avaya Labs, wadler@avaya.com

Abstract. The second-order polymorphic lambda calculus, F2, was independently discovered by Girard and Reynolds. Girard additionally proved a representation theorem: every function on natural numbers that can be proved total in second-order intuitionistic propositional logic, P2, can be represented in F2. Reynolds additionally proved an abstraction theorem: for a suitable notion of logical relation, every term in F2 takes related arguments into related results. We observe that the essence of Girard's result is a projection from P2 into F2, and that the essence of Reynolds's result is an embedding of F2 into P2, and that the Reynolds embedding followed by the Girard projection is the identity. The Girard projection discards all first-order quantifiers, so it seems unreasonable to expect that the Girard projection followed by the Reynolds embedding should also be the identity. However, we show that in the presence of Reynolds's parametricity property that this is indeed the case, for propositions corresponding to inductive definitions of naturals, products, sums, and fixpoint types.

1 Introduction

Double-barreled names in science may be special for two reasons: some belong to ideas so subtle that they required two collaborators to develop; and some belong to ideas so sublime that they possess two independent discoverers. The Curry-Howard isomorphism is an idea of the first sort that guarantees the existence of ideas of the second sort, such as the Hindley-Milner type system and the Girard-Reynolds polymorphic lambda calculus.

The Curry-Howard isomorphism consists of a correspondence between a logical calculus and a computational calculus. Each logical formula corresponds to a computational type, each logical proof corresponds to a computational term, and reduction of proofs corresponds to reductions of terms. This last point means that it is not just formulas and proofs that are preserved by the correspondence, but the structure between them as well; hence we have no mere bijection but a true isomorphism.

Curry formulated this principle for combinatory logic and combinator terms [CF58]. Howard observed that it also applies to intuitionistic propositional logic and simply-typed lambda terms [How80]. The same idea extends to a correspondence between first-order intuitionistic logic with propositional variables and simply-typed lambda calculus with type variables, which explains why the logician Hindley and the computer scientist Milner independently discovered the Hindley-Milner type system [Hin69,Mil78,DM82]. It also extends to a
correspondence between second-order intuitionistic logic with quantifiers over proposition variables and second-order typed lambda calculus with quantifiers over type variables, which explains why the logician Girard and the computer scientist Reynolds independently discovered the polymorphic lambda calculus \[\text{Gir72, Rey74}.\]

Girard and Reynolds each made additional discoveries about the calculus that bears their name, henceforth referred to as F2. Girard proved a \textit{representation theorem}: every function on natural numbers that can be proved total in second-order predicate calculus P2 (with both first- and second-order quantifiers) can be represented in F2 (using second-order quantifiers only). Reynolds proved an \textit{abstraction theorem}: for a suitable notion of logical relation, every term in F2 takes related arguments into related results [Rey83].

The calculus P2 is larger than the image under the Curry-Howard isomorphism of F2: the former has first-order terms (we will take these to be terms of untyped lambda calculus) and both first- and second-order quantifiers, while the latter has second-order quantifiers only. Nonetheless, the essence of Girard’s result is a projection from P2 onto F2 that is similar to the Curry-Howard isomorphism, in that it takes formulas to types and proofs to terms, but differs in that it erases all information about first-order terms and first-order quantifiers. This mapping also preserves reductions, so it is no mere surjection but a true homomorphism.

Reynolds’s result traditionally concerns binary relations, but it extends to other notions of relation, including a degenerate unary case. In the unary version, the essence of Reynolds’s result is an embedding from F2 into P2 that is similar to the Curry-Howard isomorphism, in that it takes types to formulas and proofs to terms, but differs in that it adds information about first-order quantifiers and first-order terms. This mapping also preserves reductions, so it is no mere injection but a true homomorphism. Furthermore, the result on binary relations can be recovered from the result on unary relations by a doubling operation, an embedding from P2 into P2 that takes formulas into formulas, proofs into proofs, and preserves reductions.

Strachey distinguished two types of polymorphism, where the meaning of a term depends upon a type [Str67]. In \textit{parametric} polymorphism, the meaning of the term varies uniformly with the type (an example is the length function), while in \textit{ad hoc} polymorphism, the meaning of the term at different types may not be related (an example is plus, which may have quite different meanings on integers, floats, and strings). Reynolds introduced a \textit{parametricity} condition to capture a semantic notion corresponding to Strachey’s parametric polymorphism. One consequence of the parametricity condition is the Identity Extension Lemma, which asserts that the relation corresponding to a type is the identity relation, so long as the relation corresponding to any free type variable is also taken to be the identity relation.

The Reynolds embedding followed by the Girard projection is the identity. Remarkably, I can find no place in the literature where this is remarked!