2 Real Spectral Triples and Charge Conjugation

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You may think of a real structure on a spectral triple as a generalization of the charge conjugation operator acting on the spinor bundle over a spin manifold. The charge conjugation operator is, in fact, an important example and will be considered in detail below. Almost everything in this section is due to Alain Connes [39].

2.1 Real Structures on Even Spectral Triples

**Definition 1 ([39])**. Let \((A, H, D)\) be an even spectral triple. A real structure of dimension \(2p \mod 8\) on \((A, H, D)\) is a conjugate linear isometry \(J : H \to H\) satisfying:

a) \(JD = DJ\), \(J^2 = \epsilon\), \(J\gamma = \epsilon'\gamma J\);

b) for any \(a \in A\), the operators \(a\) and \([D, a]\) commute with \(JAJ^*\).

\(\epsilon, \epsilon' \in \{\pm 1\}\) depend on \(d = 2p \mod 8\) according to the following table:

<table>
<thead>
<tr>
<th>(d)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\epsilon)</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>(\epsilon')</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

\((A, H, D, J)\) is called a **real spectral triple of dimension** \(2p \mod 8\).

Notice that \((\lambda J)^2 = |\lambda|^2 J\) for all \(\lambda \in \mathbb{C}\). Hence if \(J\) is a real structure of dimension \(2p\), then so is \(\lambda J\) for all \(\lambda \in \mathbb{C}\), \(|\lambda| = 1\).

The crucial part of Def. 1 is condition b). Since \(A\) commutes with \(JAJ^*\), we can turn \(H\) into a bimodule over \(A\) by putting

\[ a\xi b := aJb^*J^*(\xi) \quad \forall a, b \in A, \; \xi \in H. \]

Using that \(J\) is an isometry (that is, \(J^*J = 1\)) and that \(JAJ^*\) commutes with \(A\), one verifies easily the conditions for a bimodule. In the application to the standard model, this bimodule structure makes sense of \(u\xi u^*\) for \(u\) in the gauge group \(U(A)\) and thus allows us to define the "adjoint" representation of the gauge group \(U(A)\) on \(H\).

Condition a) is related to the notion of a "real" algebra. Let us first not worry about the signs \(\epsilon, \epsilon'\). Ignoring the dimension, we may replace the conditions \(J^2 = \epsilon\) and \(J\gamma = \epsilon'\gamma J\) by \(J^2 = \pm 1\) and \(J\gamma = \pm \gamma J\) because any pair \((\epsilon, \epsilon') \in \{\pm 1\} \times \{\pm 1\}\) occurs for a unique \(d \in \{0, 2, 4, 6\}\).

**Definition 2.** Let \(B\) be a graded \(*\)-algebra over \(\mathbb{C}\) with grading \(x \mapsto x^\gamma\) and involution \(x \mapsto x^*\). A real structure on \(B\) is a conjugate linear homomorphism \(B \to B, x \mapsto \overline{x}\), satisfying \(\overline{x} = x, \overline{x\gamma} = (\overline{x})^\gamma\), and \(\overline{x^\gamma} = (\overline{x})^\gamma\) for all \(x \in B\).

An element \(x \in B\) is called real if \(\overline{x} = x\).

A "real" graded \(*\)-algebra is a graded \(*\)-algebra with real structure.
If $B$ is a “real” graded $*$-algebra, then the set of real elements

$$B_{\mathbb{R}} := \{ x \in B \mid \overline{x} = x \}$$

is a graded $*$-algebra over $\mathbb{R}$ such that $B \cong B_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and the conjugation on $B$ is the standard conjugation on a complexification, $x \otimes \bar{x} := x \otimes \overline{x}$. Hence we could alternatively define a “real” structure as an isomorphism $B \cong B_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ for some $\mathbb{R}$-algebra $B_{\mathbb{R}}$.

**Lemma 1.** Let $\mathcal{H}$ be a graded Hilbert space with grading $\gamma : \mathcal{H} \to \mathcal{H}$. Let $J : \mathcal{H} \to \mathcal{H}$ be a conjugate linear isometry satisfying $J^2 = \pm 1$ and $J \gamma = \pm \gamma J$. Then $\overline{x} := JxJ^*$ defines a real structure on $\mathcal{L}(\mathcal{H})$.

Conversely, if $x \mapsto \overline{x}$ is a real structure on $\mathcal{L}(\mathcal{H})$, then there is a conjugate linear isometry $J : \mathcal{H} \to \mathcal{H}$ satisfying $J^2 = \pm 1$ and $J \gamma = \pm \gamma J$ such that $\overline{x} = JxJ^*$.

In addition, $J$ is unique up to multiplication with scalars of modulus 1.

**Proof.** Let $J$ be a conjugate linear isometry satisfying $J^2 = \pm 1$ and $J \gamma = \pm \gamma J$. Then $\overline{x} := JxJ^*$ is conjugate linear and $*$-preserving. Since $J$ is an isometry, $\overline{x} \cdot \overline{y} = JxJ^*JyJ^* = J(xy)J^* = \overline{x} \cdot \overline{y}$, that is, conjugation is multiplicative. Since $J^2 = \pm 1$, we have $\overline{\overline{x}} = Jx(J^*)^2 = (\pm 1)^2x = x$ for all $x \in \mathcal{L}(\mathcal{H})$, that is, conjugation is an involution. Finally, $J \gamma = \pm \gamma J$ implies $\gamma J^* = \pm J^* \gamma$ and thus

$$\overline{\overline{x}} = \overline{\overline{x}} = J \gamma x J^* \gamma = (\pm 1)^2 \gamma Jx J^* \gamma = \gamma \overline{x} \gamma = (\overline{x})^\gamma$$

for all $x \in \mathcal{L}(\mathcal{H})$. Thus $x \mapsto \overline{x}$ is a real structure on the graded $*$-algebra $\mathcal{L}(\mathcal{H})$ in the sense of Def. 2.

Conversely, let $x \mapsto \overline{x}$ be a real structure on $\mathcal{L}(\mathcal{H})$. This “conjugate automorphism” is necessarily inner in the sense that there is a conjugate linear isometry $J : \mathcal{H} \to \mathcal{H}$ such that $\overline{x} = Jx J^*$. To see this, pick any conjugate linear isometry $J' : \mathcal{H} \to \mathcal{H}$ and consider $x \mapsto J' \overline{x} J'^*$. This is a $*$-automorphism of $\mathcal{L}(\mathcal{H})$ and therefore inner.

Since $\overline{\overline{x}} = x$, we have $J^2x(J^*)^* = x$ for all $x \in \mathcal{L}(\mathcal{H})$. Thus $J^2$ is in the center of $\mathcal{L}(\mathcal{H})$. Therefore, $J^2 = \lambda$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Since $\lambda \cdot \text{id} = J^2 = J(J^2)J^* = \overline{\lambda} \cdot \text{id} = \overline{\lambda} \cdot \text{id}$, we automatically have $\lambda \in \{-1, +1\}$. Thus $J^2 = \pm 1$.

Since $\overline{\overline{x}} = (\overline{x})^\gamma$, we have $J \gamma x J^* = \gamma Jx J^* \gamma$ for all $x \in \mathcal{L}(\mathcal{H})$. Equivalently, $(J^* \gamma J \gamma)x(J^* \gamma J \gamma)^*$ for all $x \in \mathcal{L}(\mathcal{H})$, that is, $J^* \gamma J \gamma$ is in the center of $\mathcal{L}(\mathcal{H})$. Thus $J \gamma = \lambda \gamma J$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Since $J = J^2 = \lambda \gamma J \gamma = \lambda^2 \gamma J = \lambda^2 J$, it follows that $\lambda \in \{-1, +1\}$, that is, $J \gamma = \pm \gamma J$ as desired.

Finally, conjugate linear isometries $J'$ and $J$ give rise to the same real structure on $\mathcal{L}(\mathcal{H})$ iff $J' J^*$ is in the center of $\mathcal{L}(\mathcal{H})$, that is, $J' = \lambda J$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Thus $J$ is determined uniquely up to multiplication by scalars of modulus 1.