On Convex Decompositions of Points

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Abstract. Given a planar point set in general position, $S$, we seek a partition of the points into convex cells, such that the union of the cells forms a simple polygon, $P$, and every point from $S$ is on the boundary of $P$. Let $f(S)$ denote the minimum number of cells in such a partition of $S$. Let $F(n)$ be defined as the maximum value of $f(S)$ when $S$ has $n$ points. In this paper we show that $[(n-1)/4] \leq F(n) \leq [(3n-2)/5]$.

1 Introduction

Partitions of point sets into convex subsets is a ubiquitous problem in discrete geometry. The domain is a finite set of points in the plane, which we will usually denote by $S$. The points are assumed to be in general position, that is, no three points on a line. A subset of $S$ that are the vertices of a convex \textit{k-gon} is a \textit{convex subset}. A convex subset of $S$ with no points of $S$ in its open interior is called an \textit{empty convex subset}. The landmark paper of Erdős and Szekeres [3] asks for the value of the smallest integer $A(k)$ such that any set of $A(k)$ points contains a convex subset of size $k$. Subsequently a similar question is asked by Erdős in [2] for the value of the smallest integer $B(k)$ such that any set of $B(k)$ points contains an empty convex subset of size $k$. Values for $B(k)$ are known for all values of $k$ except $k = 6$. In [2] it is shown that $B(3) = 4$, and $B(4) = 5$. In [4] it is shown that $B(5) = 10$, in fact, Figure 1 shows a 9 point set with no empty convex pentagons. Horton [5] gives a construction showing that $B(7)$ is not finite, that is, there are arbitrarily many points with no empty convex 7-gons. The value of $B(6)$ is not known, and this remains a tantalizing long outstanding open problem. Some experimental results showing that $B(6) > 20$ as well as an algorithm for computing maximum empty convex subsets were first shown in [1]. A 26 point set with no empty hexagons is given in [6]. Some combinatorial results on partitions of point sets in two and three dimensions, are presented in [7] and [8].

In this paper we consider the following variation on the convex partition theme. Given a set of points $S$ we want to partition $S$ into empty convex subsets such that the union of the subsets form a single simple polygon $P$, and every point in $S$ is on the boundary of $P$. Here, we call such an empty convex subset of \textit{k-gon} in $P$ a \textit{k-cell}. Given $S$, let $f(S)$ represent the minimum number of cells
obtained in such a partition of $S$. Let $F(n)$ denote the maximum value of $f(S)$ over all sets $S$ with $n$ points.

For example, a 9 point set $S$ in Figure 1 gives $f(S) = 4$. For any simple polygon with order $n$, since we can always triangulate its interior, the trivial upper bound of $F(n)$ is $n - 2$.

![Fig. 1. The construction with 9 points, partitioned into three 4-cells and one 3-cell.](image)

In the next section we prove the following theorem.

**Theorem 1.** \[
\left\lfloor \frac{n-1}{4} \right\rfloor \leq F(n) \leq \left\lfloor \frac{3n-2}{5} \right\rfloor
\]

2 Upper and lower bounds

As was shown in [5] there exists sets $n = 2^k$ with no empty convex hexagons giving the lower bound $n/4$. We obtain here the lower bound for any integer $n$.

![Fig. 2. An example to illustrate the lower bound](image)