On Paths in a Complete Bipartite Geometric Graph

Atsushi Kaneko\textsuperscript{1} and M. Kano\textsuperscript{2}

\textsuperscript{1} Department of Computer Science and Communication Engineering, Kogakuin University, Shinjuku-ku, Tokyo 1563-8677, Japan
\textsuperscript{2} Department of Computer and Information Sciences, Ibaraki University, Hitachi 316-8511, Japan

Abstract. Let $A$ and $B$ be two disjoint sets of points in the plane such that no three points of $A \cup B$ are collinear, and let $n$ be the number of points in $A$. A geometric complete bipartite graph $K(A, B)$ is a complete bipartite graph with partite sets $A$ and $B$ which is drawn in the plane such that each edge of $K(A, B)$ is a straight-line segment. We prove that (i) If $|B| \geq (n + 1)(2n - 4) + 1$, then the geometric complete bipartite graph $K(A, B)$ contains a path that passes through all the points in $A$ and has no crossings; and (ii) There exists a configuration of $A \cup B$ with $|B| = \frac{n^2}{4} + \frac{n}{2} - 1$ such that in $K(A, B)$ every path containing the set $A$ has at least one crossing.

1 Introduction

Let $G$ be a finite graph without loops or multiple edges. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of $G$, respectively. For a vertex $v$ of $G$, we denote by $\deg_G(v)$ the degree of $v$ in $G$. For a set $X$, we denote by $|X|$ the cardinality of $X$. A geometric graph $G = (V(G), E(G))$ is a graph drawn in the plane such that $V(G)$ is a set of points in the plane, no three of which are collinear, and $E(G)$ is a set of (possibly crossing) straight-line segments whose endpoints belong to $V(G)$. If a geometric graph $G$ is a complete bipartite graph with partite sets $A$ and $B$, i.e., $V(G) = A \cup B$, then $G$ is denoted by $K(A, B)$, which may be called a geometric complete bipartite graph.

In 1996, M. Abellanas, J. García, G. Hernández, M. Noy and P. Ramos \cite{1} showed the following result.

Theorem A (Abellanas et al. \cite{1}) Let $A$ and $B$ be two disjoint sets of points in the plane such that $|A| = |B|$ and no three points of $A \cup B$ are collinear. Then the geometric complete bipartite graph $K(A, B)$ contains a spanning tree $T$ without crossings such that the maximum degree of $T$ is $O(\log |A|)$.

In 1999, Kaneko \cite{3} improved their result and proved the following theorem.

Theorem B (Kaneko \cite{3}) Let $A$ and $B$ be two disjoint sets of points in the plane such that $|A| = |B|$ and no three points of $A \cup B$ are collinear. Then the geometric complete bipartite graph $K(A, B)$ contains a spanning tree $T$ without crossings such that the maximum degree of $T$ is at most 3.
It is well-known that under the same condition in Theorem B, there are configurations of $A \cup B$ such that $K(A, B)$ does not contain a Hamiltonian path without crossings [2]. Note that the upper bound of the number of crossings of Hamiltonian cycles in $K(A, B)$ is given in [4]. So we are led to the following problem. Given two disjoint sets $A$ and $B$ of points in the plane such that no three points of $A \cup B$ are collinear, if $|B|$ is large compared with $|A|$, then does $K(A, B)$ contain a path $P$ without crossings such that $V(P)$ contains the set $A$? The answer to the above question is in the affirmative, as we shall see now. We prove the following theorem.

**Theorem 1.** Let $A$ and $B$ be two disjoint sets of points in the plane such that no three points of $A \cup B$ are collinear, and let $n$ be the number of points in $A$.

(i) If $|B| \geq (n + 1)(2n - 4) + 1$, then the geometric complete bipartite graph $K(A, B)$ contains a path $P$ without crossings such that $V(P)$ contains the set $A$.

(ii) There exists a configuration of $A \cup B$ with $|B| = \frac{n^2}{16} + \frac{n}{2} - 1$ such that in $K(A, B)$ every path containing the set $A$ has at least one crossing.

In order to prove Theorem 1, we need some notation and definitions. For a set $X$ of points in the plane, we denote by $\text{conv}(X)$ the convex hull of $X$. The boundary of $\text{conv}(X)$ is a polygon whose segments and extremes are called the edges and the vertices of $\text{conv}(X)$, respectively. For two points $x$ and $y$ in the plane, we denote by $xy$ the straight line segment joining $x$ to $y$, which may be an edge of a geometric graph containing both $x$ and $y$ as its vertices. Let $A$ be a set of points in the plane, let $y$ be a vertex of $\text{conv}(A)$ and let $x$ be a point exterior to $\text{conv}(A)$. Then we say that $x$ sees $y$ on $\text{conv}(A)$ if the line segment $xy$ intersects $\text{conv}(A)$ only at $y$.

**Lemma 1.** Let $R$ and $S$ be disjoint sets of points in the plane with $|R| \geq |S|$ such that no three points of $R \cup S$ are collinear. Suppose that there exists a line in the plane that separates $R$ and $S$. Let $x$ and $y$ be two vertices of $\text{conv}(R \cup S)$ such that $x \in S$, $y \in R$, and $xy$ is an edge of $\text{conv}(R \cup S)$. Then in $K(R, S)$, there exists a path $P$ without crossings such that

(i) the vertex $x$ is an end of $P$, and

(ii) $P$ passes through all the points in $A$.

**Proof.** We prove the lemma by induction on $|R \cup S|$. If $|S| = 1$ or $|S| = 2$, then the lemma follows immediately, and so we may assume $|R| \geq |S| \geq 3$.

Let $x_1$ be the vertex of $\text{conv}(R \cup S)$ such that $x_1 \in S$ and $xx_1$ is an edge of $\text{conv}(R \cup S)$ (see Figure 1(b)).

Then we can find two points $z_1 \in S - \{x\}$ and $z \in R$ such that $x$ can see both $z_1$ and $z$, and $z_1z$ is an edge of $\text{conv}(R \cup S - \{x\})$ (see Figure 1(b)). Note that it may occur that $z_1 = x_1$ and/or $z = y$. Similarly, we can find two more points $w_1 \in S - \{x\}$ and $w \in R - \{z\}$ such that $z$ can see both $w_1$ and $w$, and $w_1w$ is an edge of $\text{conv}(R \cup S - \{x, z\})$ (see Figure 1(b)). Note that it may occur that $w_1 = z_1$ (and/or $w = y$ if $z \neq y$).

We now apply the inductive hypothesis to $S - \{x\}, R - \{z\}, w_1$, and $w$. Then there exists a path $P'$ in $K(S - \{x\}, R - \{z\})$ without crossings that starts with