On Finding Maximum-Cardinality Symmetric Subsets

Peter Brass

Free University Berlin, Institute of Computer Science,
Takustrasse 9, D-14195 Berlin, Germany,
brass@inf.fu-berlin.de

Abstract. In this paper I study the complexity of the problem of finding a symmetric subset of maximum cardinality among \( n \) point in the plane, or in three-dimensional space, and some related problems like the largest repeated or \( k \)-fold repeated subsets. For the maximum-cardinality symmetric subset problem in the plane I show a connection to the maximum number of isosceles triangles among \( n \) points in the plane; if this number is denoted by \( I(n) \) this gives an algorithm of complexity 
\[
O\left( (n^2 + I(n)) \log n \right) = O(n^{2.136+\epsilon} \log n).
\]

1 Results

Finding the symmetries of a set of \( n \) points, or more general testing two sets for congruence and finding all congruence mappings between them, is an old and well-studied problem [3, 2, 4, 7–9, 11], which is solved satisfactorily in dimensions two and three \( (O(n \log n)) \) and remains an interesting problem in higher dimensions. There are at least two ways to make the problem more realistic: allowing for errors in the points (Hausdorff-approximate symmetry) and for errors in the sets (large symmetric subsets). The Hausdorff-approximate symmetry recognition, however, is NP-complete [12], whereas the identification of large symmetric parts in the exact model leads to interesting problems, which are related to combinatorial geometry in a way already apparent in several other exact point pattern matching problems [1, 6].

There are several ways to formalize the notion of ‘large symmetric parts’ contained in a point set. The most obvious is to determine the largest-cardinality subset with a nontrivial symmetry (Figure 1 shows a set, the largest-cardinality symmetric subset, and another symmetric subset). For this problem Eades [8] gave an \( O(n^4 \log n) \)-algorithm.

Let \( I(n) \) denote the maximum number of isosceles triangles that can occur among \( n \) points in the plane. Then holds

**Theorem 1.** The largest-cardinality symmetric subset of a set of \( n \) points in the plane can be determined in \( O\left( (n^2 + I(n)) \log n \right) \) time.

A classical and very simple bound is \( I(n) = O(n^{2+\frac{1}{3}}) \), obtained by counting incidences of points and mid-perpendiculars [15], with a lower bound \( I(n) =
\( \Omega(n^2 \log n) \) given by the integer lattice. The upper bound was recently improved to \( I(n) = O(n^{2.136+\varepsilon}) \) for every positive \( \varepsilon \) [16]. This implies

**Corollary 1.** The largest-cardinality symmetric subset of a set of \( n \) points in the plane can be determined in \( O(n^{2.136+\varepsilon}) \) time, for every positive \( \varepsilon \).

![Figure 1](image)

Our algorithm lists as an intermediate result all regular polygons contained in that set. It is remarkable that this can indeed be done in that time, since for each fixed \( k \) there are sets of \( n \) points containing \( c_k n^2 \) regular \( k \)-gons [10, 14].

A different formalization is to ask for the largest subset \( Y \) of the given set \( X \) that is repeated: there is a nontrivial motion \( \varphi \) with \( Y \subset X \) and \( \varphi(Y) \subset X \); or that is \( r \)-fold repeated: \( Y \subset X, \varphi(Y) \subset X, \ldots, \varphi^{r-1}(Y) \subset X \). (Figure 2 shows a set, a 8-fold repeated subset, and a once repeated subset.) This notion captures parts of some bigger symmetric structure, e.g. some finite part of an infinite frieze group symmetry. The special case of equidistant collinear rows of points (\( \varphi \) a translation, \( Y \) only one point) was also studied previously [5, 13, 17].

**Theorem 2.** The largest \( r \)-fold repeated subset of a set of \( n \) points in the plane can be determined in \( O(n^{3+\varepsilon} \log n) \) for \( r = 1 \) and \( O(n^{2+\varepsilon} \log n) \) for \( r \geq 2 \).

![Figure 2](image)

Essentially the same algorithm works for both problems also in three-dimensional space (but not in higher dimensions) where we get a time bound \( O(n^3 \log n) \).

## 2 The basic algorithm

In all the above cases symmetries by reflections are simple, and can be enumerated trivially in \( O(n^2 \log n) \) time, since we have to look only at the \( \binom{n}{2} \) possible pairs of points that can be exchanged by a reflection, and see which reflection line occurs most frequently. So in the following we will only look for rotation symmetries. Also, the algorithms for the different problems are almost the same (with an important difference only in the case of finding one time repeated sets), so we will give only the first, and state the necessary modifications later.