Market Split and Basis Reduction: Towards a Solution of the Cornujoëls-Dawande Instances

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Abstract. At the IPCO VI conference Cornujoëls and Dawande proposed a set of 0-1 linear programming instances that proved to be very hard to solve by traditional methods, and in particular by linear programming based branch-and-bound. They offered these market split instances as a challenge to the integer programming community. The market split problem can be formulated as a system of linear diophantine equations in 0-1 variables.

In our study we use the algorithm of Aardal, Hurkens, and Lenstra (1998) based on lattice basis reduction. This algorithm is not restricted to deal with market split instances only but is a general method for solving systems of linear diophantine equations with bounds on the variables.

We show computational results from solving both feasibility and optimization versions of the market split instances with up to 7 equations and 60 variables, and discuss various branching strategies and their effect on the number of nodes enumerated. To our knowledge, the largest feasibility and optimization instances solved before have 6 equations and 50 variables, and 4 equations and 30 variables respectively.

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We also present a probabilistic analysis describing how to compute the probability of generating infeasible market split instances. The formula used by Cornujoëls and Dawande tends to produce relatively many feasible instances for sizes larger than 5 equations and 40 variables.

1 Introduction and Problem Description

The feasibility version of the market split problem is described as follows. A company with two divisions supplies retailers with several products. The goal is to allocate each retailer to one of the divisions such that division 1 controls $100c_i\%$, 0 ≤ $c_i$ ≤ 1, of the market for product $i$, and division 2 controls $(100 - 100c_i)\%$. There are $n$ retailers and $m \leq n$ products. Let $a_{ij}$ be the demand of retailer $j$ for product $i$ and let $d_i$ be determined as $\lfloor c_id_i \rfloor$, where $d_i$ is the total amount of product $i$ that is supplied to the retailers. The decision variable $x_j$ takes value 1 if retailer $j$ is allocated to division 1 and 0 otherwise. The question is: “does there exist an allocation of the retailers to the divisions such that the desired market split is obtained?” One can formulate this problem mathematically as follows:

\[ \text{FP}: \text{does there exist a vector } \mathbf{x} \in \mathbb{Z}^n : \mathbf{Ax} = \mathbf{d}, \ 0 \leq \mathbf{x} \leq 1? \] (1)

Let $X = \{ \mathbf{x} \in \{0,1\}^n : \sum_{j=1}^{n} a_{ij}x_j = d_i, \ 1 \leq i \leq m \}$. Problem FP is NP-hard due to the bounds on the variables. The algorithm that we use was developed for the more general problem:

\[ \text{does there exist a vector } \mathbf{x} \in \mathbb{Z}^n : \mathbf{Ax} = \mathbf{d}, \ -1 \leq \mathbf{x} \leq \mathbf{u}? \] (2)

We assume that $\mathbf{A}$ is an integral $m \times n$ matrix, where $m \leq n$, $\mathbf{d}$ is an integral $m$-vector, and $\mathbf{l}$ and $\mathbf{u}$ are integral $n$-vectors. We denote the $i$th row of the matrix $\mathbf{A}$ by $a_i$. Without loss of generality we assume that $\gcd(a_{11}, a_{12}, \ldots, a_{1m}) = 1$ for $1 \leq i \leq m$, and that $\mathbf{A}$ has full row rank.

In the optimization version of the market split problem we want to find the minimum slack, positive or negative, that needs to be added to the diophantine equations in order to make the system feasible:

\[ \text{OPT}: \min \left\{ \sum_{i=1}^{m} (s_i + w_i) : (\mathbf{x}, \mathbf{s}, \mathbf{w}) \in X^S \right\}, \] (3)

where $X^S = \{ (\mathbf{x}, \mathbf{s}, \mathbf{w}) : \sum_{j=1}^{n} a_{ij}x_j + s_i - w_i = d_i, \ 1 \leq i \leq m, \ \mathbf{x} \in \{0,1\}^n, \ \mathbf{s}, \mathbf{w} \in \mathbb{Z}^n \}$.

The Cornujoëls-Dawande instances [2] of the market split problem were generated such that they were hard for linear programming based branch-and-bound, and they appear to be hard for several other methods as well. The input was generated as follows. Let $n = 10(m - 1)$ and let the coefficients $a_{ij}$ be integer numbers drawn uniformly and independently from the interval $[0, D - 1]$, where $D = 100$. The right-hand-side coefficients are computed as