Randomized Lower Bounds for Online Path Coloring *

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Abstract. We study the power of randomization in the design of online graph coloring algorithms. No specific network topology for which randomized online algorithms perform substantially better than deterministic algorithms is known until now. We present randomized lower bounds for online coloring of some well studied network topologies.

We show that no randomized algorithm for online coloring of interval graphs achieves a competitive ratio strictly better than the best known deterministic algorithm [KT81].

We also present a first lower bound on the competitive ratio of randomized algorithms for path coloring on tree networks, then answering an open question posed in [BEY98]. We prove an \( \Omega(\log \Delta) \) lower bound for trees of diameter \( \Delta = O(\log n) \) that compares with the known \( O(\Delta) \)-competitive deterministic algorithm for the problem, then still leaving open the question if randomization helps for this specific topology.

1 Introduction

In this paper we present randomized lower bounds for a class of online graph coloring problems. The input instance to an online graph coloring problem is a sequence \( \sigma = \{v_1, \ldots, v_m\} \) of vertices of a graph. The algorithm must color the vertices of the graph following the order of the sequence. When the color is assigned to vertex \( v_i \), the algorithm can only see the graph induced by vertices \( \{v_1, \ldots, v_i\} \). The goal of a graph coloring algorithm is to use as few colors as possible under the constraint that adjacent vertices receive different colors.

Online graph coloring problems have been studied by several authors [KT91,HS92,LST89,Vis90]. The study of online graph coloring has actually been started even before the notion of competitive analysis of online algorithms was

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introduced [ST85]. Kierstead and Trotter [KT81] in 1981 considered the online coloring problem for interval graphs. Every vertex of an interval graph is associated with an interval of the line. Two vertices are adjacent if the two corresponding intervals are intersecting. Since interval graphs are perfect graphs [Go80], they have chromatic number \( \chi \) equal to the maximum clique size \( \omega \), i.e., the maximum number of intervals overlapping at the same point of the line.

In [KT81] it is presented a deterministic online algorithm that colors an interval graph of chromatic number \( \omega \) with \( 3\omega - 2 \) colors. They also prove that the \( 3\omega - 2 \) bound is tight: for every deterministic algorithm there exists an input sequence where the algorithm uses at least \( 3\omega - 2 \) colors.

The online interval graph coloring problem has a natural extension to trees. Every vertex of the graph, called the intersection graph, is in this case associated with a path on a tree network. Two vertices are adjacent in the graph if the two corresponding paths are intersecting. This problem has recently received a growing attention due to its application to wavelength assignment in optical networks [RU94,BL97,GSR96].

Several authors show an \( O(\Delta) \) competitive deterministic algorithm for the problem of coloring online paths on a tree network (see for instance [BL97,GSR96]), where \( \Delta \) is the diameter of the graph. Bartal and Leonardi [BL97] also show an almost matching \( \Omega(\Delta/\log \Delta) \) deterministic lower bound on a tree of diameter \( \Delta = O(\log n) \), where \( n \) is the number of vertices of the graph.

In this paper we present the first randomized lower bounds on the competitive ratio of randomized algorithms for online interval graph coloring and online coloring of paths on tree networks.

Randomized algorithms for online problems [BDBK+90] have often been proved to achieve competitive ratios that are strictly better than deterministic online algorithms. The competitive ratio of a randomized algorithm against an oblivious adversary is defined as the maximum over all the input sequences of the ratio between the expected online cost and the optimal offline cost. The input sequence for a given algorithm is generated by the oblivious adversary without knowledge of the random choices of the algorithm.

However, for no network topology it is known a randomized online coloring algorithm that achieves a substantially better competitive ratio that the best deterministic algorithm for the problem. The first result we present in the paper is along this direction.

We present the first randomized lower bound, up to our best knowledge, for online coloring of interval graphs. We show that any randomized algorithm uses an expected number of colors equal to \( 3\omega - 2 - o(1/\omega) \) for an interval graph of maximum clique size equal to \( \omega \), thus proving that randomization does not basically improve upon the best deterministic algorithm of [KT81].

Our second result is the first randomized \( \Omega(\log \Delta) \) lower bound for online coloring of paths on a tree network of diameter \( \Delta = O(\log n) \), then answering an open question of Borodin and El-Yaniv [BEY98]. There is still a substantial gap