Accuracy of Node-Based Solutions on Irregular Meshes
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Introduction

This paper presents an analysis of the order-of-accuracy of discrete steady-state solutions obtained using Jameson’s node-based time-marching algorithm [1]. Although this algorithm is usually used in the aeronautical community to solve the Euler or Navier-Stokes equations, in this paper it will be assumed for simplicity that the hyperbolic equation being solved is linear and has constant coefficient matrices:

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} + B \frac{\partial U}{\partial y} = 0 \quad (1)$$

Attention will be limited to irregular computational grids with triangular cells, since these are the grids for which the order of accuracy is most uncertain. Roe has proved [2] that the truncation error of the discrete steady state operator is only first order in these situations, but numerical results presented in a companion paper by Lindquist [3] show that second order accuracy can be achieved if the numerical smoothing is constructed sufficiently carefully. The aim of this paper is to explain this result, but the analysis is not sufficiently rigorous to be considered a proof.

Analysis of local truncation error

If \(U(x, y)\) is an analytic solution of the steady equation

$$LU(x, y) \equiv A \frac{\partial U}{\partial x} + B \frac{\partial U}{\partial y} = 0, \quad (2)$$

then integrating over an arbitrary control volume \(\Omega\) gives

$$\iint_{\Omega} A \frac{\partial U}{\partial x} + B \frac{\partial U}{\partial y} \, dx \, dy = \oint_{\partial \Omega} (AU \, dy - BU \, dx) = 0 \quad (3)$$

The control volume \(\Omega\) for the discrete operator is the union of the triangular cells around one node, and the discrete steady state equation is

$$L^h U^h = \frac{1}{\text{Vol}(\Omega)} \sum_{\partial \Omega} (A\bar{U} \, dy - B\bar{U} \, dx) = 0 \quad (4)$$

with \((\Delta x, \Delta y)^T\) being the vector face length in the counter-clockwise direction, and \(\bar{U}\) being the average value of \(U\) on the face obtained by a simple arithmetic average of the values at the two end nodes.

The truncation error \(T^h\) is obtained by applying the discrete operator \(L^h\) to the analytic solution \(U\).

$$T^h \equiv L^h U = \frac{1}{\text{Vol}(\Omega)} \left( \sum_{\partial \Omega} (A\bar{U} \, dy - B\bar{U} \, dx) - \oint_{\partial \Omega} (AU \, dy - BU \, dx) \right) = \frac{1}{\text{Vol}(\Omega)} \sum_{\partial \Omega} T_f \quad (5)$$

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where $T_f$ is the trapezoidal integration error on each face.

$$T_f = \frac{1}{2} A(U_1 + U_2)(y_2 - y_1) - \frac{1}{2} B(U_1 + U_2)(x_2 - x_1) - \int_1^2 (AU \, dy - BU \, dx)$$

$$= (A_n + Bn_x) \left( \frac{1}{2}(U_1 + U_2) \Delta s - \int_{x_1}^{x_2} U \, ds \right)$$

$$= \frac{1}{12} (\Delta s)^3 (A_n + Bn_x) \frac{\partial^2 U}{\partial x^2} + O \left( (\Delta s)^4 \right) \tag{6}$$

In this equation, $(n_x, n_y)^T$ is the outward pointing unit normal vector on the face, and $\Delta s$ is the length of the face.

There are two important points which arise immediately from this truncation error analysis. The first is that if $h$ is a typical cell dimension then $T_f = O(h^3)$ and $\text{Vol}(\Omega) = O(h^2)$, and hence for a general irregular mesh $T^h = O(h)$, so the local truncation error is first order. (If the mesh is sufficiently regular then, as noted by Roe, cancellation of truncation errors $T_f$ on opposing faces leads to the truncation error being second order.) The second point is that if one considers a large control volume $\Omega$ whose area is $O(1)$, and sums over all of the cells inside $\Omega$ then each of the internal faces contributes equal and opposite amounts to the truncation errors in the two cells on either side. Hence the global integrated truncation error is

$$\sum_{\partial \Omega} T_f = O(h^{-1}) \times O(h^3) = O(h^2), \tag{7}$$

since there are $O(h^{-1})$ faces along the perimeter $\partial \Omega$. These two points suggest that the truncation error has a low-frequency component, with wavelength $O(1)$, whose amplitude is second order, and a high-frequency component with wavelength $O(h)$, whose amplitude is first order. This idea is confirmed in the next section in a different manner.

**Spectral content of local truncation error**

The truncation error, $T(x, y)$, can be defined to be piecewise linear, with its value on each triangle determined by the values at the three corners. Thus,

$$T(x, y) = \sum_j T_j s_j(x, y) \tag{8}$$

where $T_j$ is the truncation error at node $j$, and $s_j(x, y)$ is the triangular shape function which varies, in a continuous piecewise-linear manner, from a value of 1 at node $j$ to 0 at its immediate neighbors, and is zero on all other triangles.

The next step is to find the spectral content of $T(x, y)$. For simplicity we now assume that the computational domain is a square of size $\pi \times \pi$, and the boundary conditions are treated perfectly so that the truncation error is zero there. Hence, $T(x, y)$ can be expressed as a sine series expansion.

$$T(x, y) = \sum_{m,n} a_{mn} \sin(mx) \sin(ny) \tag{9}$$

The sum is over positive integer values of $m$ and $n$. The amplitudes $a_{mn}$ are given by

$$a_{mn} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi T(x, y) \sin(mx) \sin(ny) \, dx \, dy. \tag{10}$$