INTRODUCTION

Our aim in this article is to describe new numerical algorithms that are well suited for the solution of the Navier-Stokes equations over large intervals of time. Although we investigate here the case of incompressible flows, the methods can be extended to other flows such as thermohydraulics and reactive flows or, as well, compressible flows.

The understanding and the numerical solution of the Navier-Stokes equations are major problems of fluid mechanics with important implications in engineering and in fundamental research: the study of compressible or incompressible flows with or without reactive, electromagnetic or thermal phenomena necessitates among other things the understanding of the Navier-Stokes equations.

On the other hand the considerable increase of the computing power due to the appearance of supercomputers has made possible the solution of numerical problems that were unthinkable a few years ago. In the case of the Navier-Stokes equations we can now consider ranges of values of the physical parameters that are close to values physically relevant (at least in two space dimensions) and we can consider the onset of turbulence. In particular we can study turbulent flows that are time dependent, the laminar stationary solution being unstable; and the computation of such flows necessitates clearly a large (theoretically infinite) time integration of the Navier-Stokes equations.

One of the first difficulties encountered in the solution of these equations and which appears even when the flow is laminar, is the treatment of the incompressibility condition

\[ \text{div } u = 0. \quad (0.1) \]

There are now several methods that are available for the treatment of (0.1) and that are well suited for computing stationary solutions and laminar flows. In particular, among the many important contributions of N.N. Yanenko to numerical analysis and computational fluid dynamics, the fractional step method (or splitting-up method) that he has introduced and contributed to develop is one of the classical available methods (see [RY][Y1,Y2]).

\[ ^1 \text{Laboratoire d'Analyse Numérique, Université Paris-Sud, Bâtiment 425, 91405 Orsay (France), and Institute for Applied Mathematics and Scientific Computing, Bloomington, Indiana (U.S.A.)} \]
The fractional step method has also laid A.J. Chorin [C2] and R. Temam [T3] to introduce the projection method. Also although this is far from the preoccupations of this article we should like to mention a recent application of the fractional step method to liquid crystal problems (see R. Cohen et al. [CHKL], R. Cohen [CO]) connected to the extension of the fractional step method to constrained optimization (see J.L. Lions and R. Temam [LT1][LT2]).

When we reach turbulent regimes new difficulties arise. In particular an essential aspect of turbulent flows is the relation/interaction between small and large eddies. All the frequencies of the spectrum, up to the Kolmogorov dissipation frequency \( \kappa_d \) interact; large eddies break into small eddies and those, in turn, feed the large eddies. Besides the usual difficulties (incompressibility, nonlinearity, large Reynolds number), a new difficulty occurring in computing high Reynolds fluid flows is the interaction of small and large eddies: small eddies are negligible at each given time but their cumulative effect is not negligible on a large interval of time. Thus a proper and economical treatment of small eddies is necessary for large time computations. The algorithms that we propose here, called nonlinear Galerkin methods, stem from recent developments in dynamical systems theory and are motivated by this preoccupation.

This article is organized as follows. In Section 1 we study the interaction of small and large eddies in a turbulent flow. In Section 2 we present the simplest nonlinear Galerkin method while Section 3 contains another (more involved) version of the method. Section 4 gives some theoretical justifications of the methods related to recent developments in the dynamical system approach to turbulence. Finally Section 5 presents the results of some spectral numerical computations based on the nonlinear Galerkin method in Section 2; these results show for a given accuracy, a significant gain in computing time (of the order of 20\% to 40\%).

CONTENTS.

1. Interaction of small and large eddies.
2. The nonlinear Galerkin method.
3. Another nonlinear Galerkin method.
4. Theoretical justification: attractors and inertial manifolds.
5. Numerical results.

1. INTERACTION OF SMALL AND LARGE EDDIES.

We consider in space dimension \( n = 2 \) or 3 the incompressible Navier-Stokes equations with density 1:

\[
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \tag{1.1}
\]

\[
\nabla \cdot u = 0. \tag{1.2}
\]