COVERING RADIUS: IMPROVING ON THE SPHERE-COVERING BOUND

Juriaan Simonis

Delft University of Technology
Faculty of Mathematics and Informatics
P.O. Box 356, 2600 AJ DELFT, HOLLAND

Abstract: Currently, the best general lower bound for the covering radius of a code is the sphere covering bound. For binary linear codes, the paper presents a new method to detect cases in which this bound is not attained.

I. INTRODUCTION

Starting with a multiset\(^1\) \(S\) of cardinality \(n\) in \(\mathbb{F}_2^r\), we define a sequence of \(r+1\) multisets \(S^\ell\) of cardinality \(\binom{n}{\ell}\):

\[
S^\ell := \begin{cases} 
\{0\} & \text{if } \ell = 0, \\
\{ \sum_{x \in U} x \mid U \subset S \land |U| = \ell \} & \text{if } \ell > 0. 
\end{cases}
\]

So \(S^\ell\) consists of all linear combinations in which exactly \(\ell\) vectors of \(S\) are involved.

Suppose that \(S\) \textbf{spans} the \(\mathbb{F}_2\)-vector space \(\mathbb{F}_2^r\), i.e. that

\[
\bigcup_{\ell=0}^{r} S^\ell \supset \mathbb{F}_2^r.
\]

Then there is a well-defined smallest integer \(t\) such that

\[
\bigcup_{\ell=0}^{t} S^\ell \supset \mathbb{F}_2^r.
\]

\((*)\)

This integer \(t\) is called the \textbf{covering radius} \(t(S)\) of \(S\) in \(\mathbb{F}_2^r\).

\(^1\)The notion of multiset will be explained in the appendix.
Remarks:

a) This notion of covering radius clearly is invariant under coordinate transformations of \( \mathbb{F}_2^r \). So we could — and perhaps should — have replaced \( \mathbb{F}_2^r \) by an arbitrary \( r \)-dimensional \( \mathbb{F}_2 \)-vector space \( V \).

b) The usual definition of covering radius can, for instance, be found in Cohen et al. [3]. Our number \( t(S) \) equals the covering radius (in their sense) of the linear code \( C(S)^\perp \), to be defined in section II.

The major problems with respect to the covering radius are as easy to formulate as they are hard to solve:

1) to determine, for all \( r \) and \( t \), the value of \( n[r,t] \), the cardinality of the smallest \( S \) in \( \mathbb{F}_2^r \) for which \( t(S) \) equals \( t \),

2) to describe these \( S \), up to linear equivalence.

Remark:

Obviously, such a minimal \( S \) does not contain the zero vector and has no multiple elements, i.e. is a proper set.

Graham and Sloane [4] have drawn up a table that gives recent information on \( n[r,t] \) for small \( r \) and \( t \). The open cases start with

- \( 16 \leq n[7,2] \leq 19 \)
- \( 23 \leq n[8,2] \leq 28 \)
- \( 15 \leq n[9,3] \leq 19 \).

One lower bound for \( n[r,t] \) is completely obvious: since the left hand side of (*) must contain at least as many vectors as the right hand side, we have