Abstract

The present work takes place in the study of infinitary behaviours for CCS-like communicating processes. A problem in that area arises from the fact that most of the abstraction morphisms we are interested in don’t commute with least fixed points. In order to offer an alternative to least fixed point semantics we present an axiomatisation of the notion of fixed point calculus within the formalism of algebraic theories. Such a calculus fixes one solution for each equation resulting from the interpretation of a set of recursive definitions in a way consistent with the free interpretation of the equations. This leads us to the notion of algebraically closed theory which stands for an algebraic theory equipped with a fixed point calculus. The rational theories by ADJ appear to be a special case of algebraically closed theories when least solutions are always chosen.

1 Introduction

Motivation for the present work are to be found in the study of infinitary behaviours of processes in CCS-like languages [Mil80]. Such languages are essentially defined by their signature $\Sigma = \bigcup_{n \in \omega} \Sigma_n$ ($\Sigma_n$ stands for the $n$-ary operator symbols). Terms built on that signature denotes finite processes, whereas infinite processes are defined recursively, using for example the classical combinator rec: for instance, if $\Sigma_0 = \{ \text{nil} \}$ and $\Sigma_1 = \{ a. - / a \in A \}$ where $A$ is a set of actions and $\Sigma_2 = \{ +, 1 \}$, then the rational expression let $\text{rec} (x = \alpha.\text{nil} + \beta. x)$ in $\gamma.x$ is a finite description of the tree obtained by unfolding infinitely many times the definition $x = \alpha.\text{nil} + \beta. x$, namely

$$\gamma.(\alpha.\text{nil} + \beta.(\alpha.\text{nil} + \beta(\alpha.\text{nil} + \beta \ldots )))$$

A model for such a language must provide, first of all, an interpretation of the operator symbols in a semantical domain $D$, supplying for each $n$-ary operator symbol $f \in \Sigma_n$ a corresponding $n$-ary operator on that domain $\delta_f : D^n \rightarrow D$, the $\Sigma$-algebra $\delta = \{ \delta_f; f \in \Sigma \}$ procures an interpretation in $D$ for every closed $\Sigma$ term. In the same way, a $\Sigma$-algebra defines implicitly for each system of recursive definitions, an associated system of equations on the domain $D$. For example, given the declaration $\text{rec}(x = \alpha.\text{nil} + \beta.x)$ there corresponds the equation $\varepsilon : \xi = \delta_\alpha(\delta_\text{nil}(\varepsilon), \delta_\beta(\varepsilon))$ on $D$. For that reason, a model must supply a solution $\varepsilon^*$ for each equation $\varepsilon$ resulting from the interpretation of a system of recursive definitions. Let the corresponding process be called a fixed point calculus. In such a model, the interpretation of the rational expression given above is now fully defined as $\delta_\gamma(\varepsilon^*)$. Then a model is a $\Sigma$-algebra + a fixed point calculus

In most denotational models, $D$ is an $\omega$-complete ordered set (i.e. $u_0 \subseteq u_1 \subseteq \ldots \subseteq u_n \subseteq \ldots$ has a least upper bound $\bigcup u_i$ in $D$) with a least element $\bot$, where operator symbols are interpreted by $\omega$-continuous operators $(f(\bigcup u_i) = \bigcup f(u_i)$ for every increasing chain $(u_i; i \in \omega)$). The Tarski’s theorem then ensures the existence of least fixed points, computed as least upper bounds of inductive chains. In the example, $\varepsilon^*$ is the least upper bound of the chain

$$\bot \subseteq \varepsilon(\bot) \subseteq \varepsilon(\varepsilon(\bot)) \subseteq \ldots \subseteq e^n(\bot) \subseteq \ldots$$

The elements of the domain $D$ are partial descriptions of objects, $\bot$ standing for the absence of information and the order $\subseteq$ expressing the fact that a partial description is less accurate than another one.

As noticed in [ADJ76] the method that consists in computing fixed points of transformations by iterating them from the least element may
work even for non $\omega$-continuous theories; theories in which such iterations provide always fixed points of transformations are called rational theories. An equivalent formulation has been given by Tiuryn using regular algebras [Tiu77]; this type of semantics is usually called least fixed point semantics.

Least fixed point semantics may be used for operational models of programs, where the objects of $D$ are, for instance, synchronization trees and in which all the actions of the processes are represented. Now, following [DG87] we intend to derive therefrom a family of models through abstraction morphisms. We may forget information because we consider that a certain kind of action is unobservable or that some distinctions that appear between processes at the operational level become irrelevant at a more abstract level. Unfortunately, as soon as we are concerned with infinite behaviours, it occurs that abstraction morphisms generally don't commute with least fixed points: least fixed point semantics are therefore no longer suitable. A simple example of that situation arises when we consider the invisible action $\tau$ of CCS which corresponds to an 'internal' action of a process. At the observational level we cannot simply erase all occurrences of internal actions, because a process which is likely to do an internal action is unstable and unstability has observable effects. For example, we don't want to identify the processes with respective synchronization trees

$$\tau.a.nil + \beta.nil \neq \alpha.nil + \beta.nil$$

since the former can perform an internal action and then become unable to do the action $\beta$. However, we don't want to distinguish two successive internal actions from one. Thus, we factor the operational model of synchronization trees by the least congruence $\approx$ for which

$$\tau.\tau.p \approx \tau.p$$

In that way, we distinguish between a finite sequence of internal actions (unstability) from an infinite sequence of internal actions (divergent process). Now the least (and actually unique) solution of the equation $x = \tau.x$ in the initial model is $\tau^\omega$, whereas all the unstable processes (s.t. $p \approx \tau.p$) are solutions in the factor model, hence the least solution is the class of $\tau.\bot$ and not the one of $\tau^\omega$.

This example shows that if we are looking for a general framework allowing us to describe the behaviours of processes at different levels of abstraction we must be able to handle fixed points that are not the least ones. There lays our motivation for going beyond the rational theories. In order to extend the ADJ approach we propose axioms for fixed point calculi within the formalism of algebraic theories. For us, a model is then an algebraically closed theory (that is to say an algebraic theory supplied with a fixed point calculus) together with a $\Sigma$-algebra structure on that theory. As for model morphisms, they must on one hand respect the interpretation of the operators (i.e. be congruences for those operators) on the other hand they must commute with the respective fixed point calculi defined on those models.

A particular role is played here by the Herbrand model, whose domain consists of the infinite rational trees built on the signature $\Sigma$. That model gives a 'minimal' interpretation of the language in that the operator symbols are freely interpreted. In the Herbrand model, each $n$-ary operator symbol $f \in \Sigma_n$ is sent to the mapping which builds the tree $f(t_0, \ldots, t_{n-1})$ from the $n$-uple of rational trees $(t_0, \ldots, t_{n-1})$; and the fixed point calculus associates to declarations the vectors of rational trees obtained by unfolding them ad infinity. The initiality of the Herbrand model means that every other model must be obtained as a morphic image of it. Otherwise stated, if two expressions stand for the same rational tree, then they must have the same interpretation in all models.

This condition is the only constraint: a fixed point calculus is a calculus which supplies one solution for each equation resulting from the interpretation of a set of recursive definitions, consistently with the free interpretation of those equations. Hence, we are looking for coherence conditions that would be strong enough to ensure the initiality of the Herbrand model and weak enough to allow for any reasonable fixed point calculus. For example, we must retrieve rational theories as a particular case of algebraically closed theories.