§1. INTRODUCTION

The formation of a distribution of cluster sizes is a common feature in a wide variety of systems. Examples include astrophysics, atmospheric physics, colloidal chemistry, polymer science and the kinetics of phase transitions in binary alloys. In this paper we discuss the mathematical theory of a model for the dynamics of cluster growth. The processes described by this model involve coagulation of clusters via binary interactions and fragmentation, a unimolecular process. The distribution of cluster sizes is determined by the competition between these processes.

If \( c_j(t) \geq 0 \), \( j=1,2,\ldots \), denotes the expected number of \( j \)-particles per unit volume, then the discrete coagulation-fragmentation equations are

\[
\frac{d}{dt} c_j = \frac{1}{2} \sum_{k=1}^{j-1} \left( a_{j-k,k} c_{j-k} c_k - b_{j-k,k} c_j \right) - \sum_{k=j+1}^\infty \left( a_{j,k} c_j c_k - b_{j,k} c_{j+k} \right) \quad (1.1)
\]

for \( j=1,2,\ldots \). The coagulation rates \( a_{j,k} \) and fragmentation rates \( b_{j,k} \) are nonnegative constants with \( a_{j,k} = a_{k,j} \) and \( b_{j,k} = b_{k,j} \). In equation (1.1) the first two terms represent the rate of change of the \( j \)-cluster due to the coalescence of smaller clusters and the breakup of the \( j \)-cluster into smaller clusters. The final two terms represent the change due to coalescence of the \( j \)-cluster with other clusters and the breakup of larger clusters into \( j \)-clusters. For a derivation of this equation and its analogue in which the cluster size is a continuous variable see [6]. The model neglects the geometrical location of clusters and spatial fluctuations in cluster density. For further information on these effects see [4, 5].
§2. SOME SPECIAL CASES

Since matter is neither destroyed nor created in the interactions described by (1.1) we expect that the density \( \rho = \sum_j c_j(t) \) is a conserved quantity. In certain circumstances however, the density is not conserved. To illustrate this and other phenomena we consider some special cases.

(a) Let \( b_{j,k} = 0 \) for all \( j \) and \( k \) so that we only consider coagulation. We further specialise to two forms of coagulation kernel:

\[
\begin{align*}
    a_{j,k} &= j^\alpha + k^\alpha \\
    a_{j,k} &= (jk)^\alpha
\end{align*}
\]

The additive form of \( a_{j,k} \) in (2.1) would arise in applications if we assumed that binary interactions occur randomly with a rate depending on the effective surface area. The multiplicative form of (2.2) would apply to situations in which bond linking was the dominant mechanism. Note that for the kernel (2.1), large-large and large-small interactions have the same order of magnitude (i.e. \( a_{j,k} \approx a_{j,j} \) for large \( j \) and small \( k \)), whereas for (2.2) large-large interactions dominate.

If \( \alpha > 1/2 \), then for the kernel (2.2), density conservation can break down in finite time [7]. This is interpreted as the appearance of an infinite cluster or gel. For the kernel (2.1), if a solution exists then density is conserved [3].

To gain some insight into the dependence of the rate of growth of clusters we use a technique due to Leyvraz and Tschudi [9] to relate solutions of (1.1) with different initial data. We first consider the kernel (2.1) so that (1.1) takes the form

\[
    \dot{c}_j = \frac{1}{2} \sum_{k=1}^{j-1} [(j-k)^\alpha + k^\alpha] c_{j-k} c_k - \sum_{k=1}^{\infty} (j^\alpha + k^\alpha) c_j c_k
\]

Let \( c_j^i \) be a solution of (2.3) with initial data \( c_j^i(0) = \delta_{j,1} \). For positive integers \( n \) define \( c_j^n(t) = (c_j^n(t)), j=1,2,... \) by

\[
    c_j^n(t) = n^{-1} c_j^{1}(n^{-1} t) \quad \text{and} \quad c_r^n(t) = 0, \ r \text{ not a multiple of } n.
\]

It is then easy to check that \( c_j^n(t) \) is a solution of (2.3) with initial data given by \( c_j^n(0) = n^{-1} \delta_{j,n} \). From (2.4) we see that the time scale for this class of solutions depends on the sign of \( \alpha - 1 \). In fact, if \( \alpha \leq 1 \), we get