1. Introduction.

In some important cases the differential equations describing the evolution of a physical system contain a small parameter $\varepsilon > 0$ and the equations depend on $\varepsilon$ in such a singular way as $\varepsilon \to 0$, that it is impossible to define limit differential equations which capture the global limit behavior of the system. In fact as $\varepsilon \to 0$ the dynamics of the system becomes more and more complex so that there is no limit dynamics and therefore no limiting differential equations. Examples of this situation are the Navier-Stokes equations (with $\varepsilon$ the inverse of the Reynold's number) and certain differential equations from the theory of phase transitions. In dealing with problems of such a singular nature, one can still try to define some limit dynamical behavior for $\varepsilon \to 0$, provided attention is restricted to special features of the dynamics which happen to possess some kind of continuous dependence in $\varepsilon$ as $\varepsilon \to 0$. The aim of this work is to show how this can be done effectively in the case of the scalar parabolic equation

\begin{align*}
\begin{cases}
  u_t = \varepsilon^2 u_{xx} + f(u), & \text{in } (0,1), \\
  u_x = 0, & \text{at } x = 0,1.
\end{cases}
\end{align*}

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This equation is the gradient system corresponding to the Liapounov functional

\[ J(u) = \int_0^1 \left( \frac{\varepsilon^2}{2} u_x^2 + F(u) \right) dx , \]

with \( f = -F' \), and it is perhaps the simplest mathematical model for the coexistence of two phases of the same substance at the transition temperature. In this context \( u \) is an "order parameter" which is related to the microscopic structure of the matter in such a way that \( u \) near -1 corresponds to one of the two phases (solid) and \( u \) near 1 corresponds to the other phase (liquid). \( F(u) \) is the specific free energy of the matter and is assumed to have two equal minima at \( u = \pm 1 \). This corresponds to the fact that by definition at the transition temperature the two phases have the same free energy. The term \( \varepsilon^2 u_x^2 / 2 \) in the expression of \( J(u) \) is added to penalize high gradients of the order parameter and therefore to model the tendency of the substance to minimize the number of interfaces separating the two phases. More information about equation (1) and more sophisticated models for phase transitions can be found in [CF] [FG] [G1] [G2].

Equation (1) is an example of the singular situation described above. It is well known that it generates a dissipative semiflow in several function spaces (for instance in \( W^{1,2} \)) and that it possesses a global attractor \( A_\varepsilon [\text{He}] [\text{H}] \). There is no global limiting system as \( \varepsilon \to 0 \), and in fact \( A_\varepsilon \) goes through a cascade of bifurcations with its structure becoming increasingly complex. For instance the number of fixed points (stationary solution of (1)) increases without bound as \( \varepsilon \to 0 \) and the same is true for the dimension of \( A_\varepsilon ([\text{He}] [\text{H}]) \). Therefore instead of considering the whole attractor, we fix our attention to the unstable manifolds of equilibria and try to analyze what happens to these sets and to the dynamics on them as \( \varepsilon \to 0 \). We show, Theorems 1 and 2 below, that, for \( \varepsilon \ll 1 \), a large part, \( \hat{W}_N \), of the unstable manifold \( W^u(\phi_N) \) of an equilibrium \( \phi_N \) of (1) with \( N \) zeros is made of