ACCURATE FINITE-DIFFERENCE METHODS FOR SOLVING NAVIER-STOKES PROBLEMS USING GREEN'S IDENTITIES

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Abstract

An investigation is presented of a method of numerical approximation to the steady-state Navier-Stokes equations in two space dimensions using the finite-difference methods of $h^4$ accuracy recently published by Dennis and Hudson (1989) in conjunction with the global, or integral, methods based on Green's identities given by Dennis and Quartapelle (1989). It is shown that uniformly $h^4$-accurate results can be obtained using this combination of methods without the necessity of making use of local approximations to the boundary vorticity. Some detailed results of computations are given for flow past a circular cylinder in the Reynolds number range 10-100 based on the diameter of the cylinder. The problem of flow in a square cavity in which one side is moved parallel to itself with constant velocity is also considered.

1. Introduction and basic method

In obtaining numerical solutions of the Navier-Stokes equations using finite-difference methods, a problem associated with all $h^4$-accurate methods is to maintain equivalent accuracy at the boundaries when boundary values must be calculated there, e.g., for the boundary vorticity in the vorticity-stream function formulation. Local methods of approximation are normally used for calculations at boundaries but Dennis and Quartapelle (1989) have reviewed various applications of Green's identities which are used to transform local boundary conditions to global conditions, termed integral conditions, over the solution domain. Since integrals are easily approximated using $h^4$-accurate methods, it is relatively easy to complete the $h^4$-accurate analogue of the equations without using local conditions.

We consider only the formulation

\[ R^{-1} \nabla^2 \zeta = (\partial \phi / \partial y)(\partial \zeta / \partial x) - (\partial \phi / \partial x)(\partial \zeta / \partial y) \] (1)
\[ \nabla^2 \phi + \zeta = 0 \] (2)

in terms of the scalar vorticity $\zeta$ and the stream function $\phi$, although the corresponding equations for unsteady flow and also the use of other dependent variables could be considered. The variables $\phi$ and $\zeta$ are dimensionless and $R$ is the Reynolds number based on some representative length and velocity.
It is supposed that the solution is required inside a region bounded by a closed curve $C$ with given conditions

$$\psi = f(x,y), \quad \frac{\partial \psi}{\partial n} = g(x,y) \text{ on } C, \quad (3a, b)$$

where $n$ is the outward normal to $C$. Equations $(1)$ and $(2)$ are solved using the Dennis and Hudson (1989) $h^4$-accurate method, assuming Dirichlet boundary conditions. The condition needed for $(2)$ is given by $(3a)$, while that needed for $(1)$ on $C$ is found from one of Green's identities. If $\psi$ is any harmonic function and $\psi$ satisfies $(2)$, we easily find that

$$\iint_{\sigma} \psi \xi d\sigma = \oint \left( \frac{\partial \psi}{\partial n} - \frac{\partial \phi}{\partial n} \right) dS, \quad (4)$$

where $\sigma$ denotes the plane region bounded by $C$ and $S$ is distance measured along $C$ in the counter-clockwise sense.

For a given function $\psi$, the right-hand side of $(4)$ is known by $(3)$ and thus $(4)$ gives a condition on $\omega$ of integral type. There is one such condition for each $\psi$ we choose and we may determine $\omega_C$ by choosing $\psi = \psi_m$ $(m = 1, 2, 3, \ldots)$ as a complete set of functions for the region interior to $C$. If $\omega$ is expressed as a series of these functions, it is easy to construct a method of utilizing the corresponding set of conditions of type $(4)$ to determine $\omega_C$ in terms of the distribution of vorticity inside $C$ (cf Dennis and Quartapelle, 1989).

Quadrature formulae of $h^4$ accuracy are used to approximate the integral on the left-hand side of $(4)$ and thus an over-all $h^4$-accurate scheme is preserved which avoids using local approximations at the boundary. This is the basic principle of the method. Of course if $C$ is a general curved boundary there may be practical problems to overcome in maintaining the $h^4$ accuracy but, nevertheless, we can find numerous practical examples in which the method is relatively straightforward. Two illustrative example are now given.

### 2. Flow past a circular cylinder

The first example is symmetrical flow past a circular cylinder treated in the usual modified polar coordinate system $(\xi, \theta)$, where

$$x = \exp(\xi) \cos \theta, \quad y = \exp(\xi) \sin \theta. \quad (5)$$

Here the domain of the solution is $\xi > 0, \ 0 < \theta < \pi$ and the contour $C$ is in fact the corresponding curve bounded by $\theta = 0, \ \theta = \pi, \ \xi = 0, \ \xi = \pi$, although in practice the infinite boundary is approximated by a finite, large enough, value of $\xi$. The harmonic functions in $(4)$ are $\psi_m = \exp(-m\xi) \sin m\theta$ and the integral conditions equivalent to $(4)$ can be reduced to (cf Dennis and Quartapelle, 1989)

$$\sum_{m=1}^{\infty} \exp \left\{ (2-m)\xi \right\} g_m(\xi) d\xi = -2\delta_{m, 1} \quad (6)$$

where

$$\omega = \sum_{m=1}^{\infty} g_m(\xi) \sin m\theta \quad (7)$$

and $\delta_{m, 1}$ is the Kronecker delta. The coefficients $g_m(\xi)$ in $(7)$ are evaluated from a numerical solution obtained using two-dimensional $h^4$-accurate finite-differences based on the scheme of Dennis and Hudson (1989).