Tensor Operator Structures in Quantum Unitary Groups*,**

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Abstract

Tensor operators acting on model spaces for the quantum group $SU_q(n)$ are defined ("q-tensor operators") and the fundamental theorem for q-tensor operators (a generalization to non-commutative co-products of the Wigner-Eckart theorem) is proved. Examples from $SU_q(2)$ are discussed.

The symmetry structure characterizing quantum groups$^{1,2,3}$ has in the past few years been found to play an important conceptual rôle in many distinct fields of physics and mathematics, from knot theory$^4$ to rational conformal field theory$^5,6$. A quantum group is, more precisely, an algebra, a deformation$^3$ of the universal enveloping algebra of an underlying classical Lie group (here a unitary group).

The great importance of symmetry techniques in quantum physics is associated with a much larger algebraic structure, the algebra of tensor operators (which includes the universal enveloping algebra as a sub-algebra). The problem we wish to pose, and resolve, is the extension to this larger algebraic structure of the symmetry associated with quantum groups.

Contrary to folk wisdom, the Hilbert space structure of quantum physics does not, a priori, guarantee the existence of a tensor operator structure (in particular, a

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version of the Wigner-Eckart theorem) for any given symmetry structure, but depends rather upon the specific way in which the symmetry is realized. In the prototypical symmetry structure in quantum physics—the quantal rotation group $SU(2)$—there are two required properties of the realization.

**Equivariance:** The action of the group generators on a tensor operator (a set of operators) realizes a linear representation defined by transformations on this set.

**Derivation:** The generators act on products as a derivation: $J_i(ab) = J_i(a)b + aJ_i(b)$.

It is consequence of these two properties that the Wigner-Clebsch-Gordan (WCG) coefficients for $SU(2)$ occur in two logically distinct ways:

(a) as coupling coefficients for the addition of angular momentum carried by kinematically independent constituent systems, and,

(b) as matrix elements (up to a rotationally invariant scale factor) of physical transition operators.

Conversely, if these two properties do not obtain, then this latter result fails.

It is not obvious that one can extend these results to quantum groups, particularly when one realizes that the derivative property corresponds to a commutative co-product, which is invalid for a general quantum group. Moreover, the equivariance condition is problematic as well. For a Lie group, equivariance is effected by the adjoint action: $ad_x(y) = [x, y]$. This cannot work for a quantum group, since the quantum group irrep corresponding to the adjoint representation is finite dimensional in contrast to the infinite number of linearly independent generators obtained under commutation.

To sort out the problems that occur it is helpful to examine the standard realization of $SU(2)$ and see how this works. In this realization (the Jordan-Schwinger mapping) the generators are:

$$J_+ = a_1\bar{a}_2, \quad J_- = a_2\bar{a}_1, \quad J_z = \frac{1}{2}(a_1\bar{a}_1 - a_2\bar{a}_2),$$  \hspace{1cm} (1a)

with

$$[\bar{a}_i, a_j] = \delta_{ij},$$  \hspace{1cm} (1b)

and all other commutators vanishing. The set of vectors $\{|jm\rangle\}$, (for $-j \leq m \leq j; \ j = 0, \frac{1}{2}, \ldots$), carrying all (unitary) irreps in this realization is given by:

$$|jm\rangle = ((j + m)!(j - m)!)^{-1/2}a_1^m a_2^{-m}|0\rangle.$$  \hspace{1cm} (2)