Bounds on the quality of approximate solutions to the Group Steiner Problem

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Abstract

The Group Steiner Problem (GSP) is a generalized version of the well known Steiner Problem. For an undirected, connected distance graph with groups of required vertices and Steiner vertices, GSP asks for a shortest connected subgraph, containing at least one vertex of each group. As the Steiner Problem is NP-hard, GSP is too, and we are interested in approximation algorithms. Efficient approximation algorithms have already been proposed, but nothing about the quality of any approximate solution is known so far. Especially for the VLSI design application of the problem, bounds on the quality of approximate solutions are of great importance.

We present a simple polynomial time approximation algorithm that computes a tree with no more than $g - 1$ times the length of a minimal tree, where $g$ is the number of required groups. In addition, we propose an extended version of this algorithm, trading quality of the solution for computation time. Here, one extreme is just our proposed approximation, and the other is an optimal solution. Moreover, we will prove the quality bound $g - 1$ for a modification of an efficient approximation algorithm proposed in the literature.

1 Introduction

Given a connected, undirected distance graph, the problem of finding a shortest connected subgraph spanning a required set of vertices, optionally using vertices of a set of so-called Steiner vertices, is commonly called Steiner Problem (SP). A variety of applications is known. Since SP is NP-hard ([K]), several approximation algorithms have been proposed in the literature. In the routing phase of the physical VLSI design process, after the placement of components on a chip, pins of different components have to be connected, with the goal of constructing a shortest possible electrical network. This minimization aims at achieving high integration of the circuit and therefore is an essential aspect in practice. Let us take into account the freedom that results from rotating and mirroring components after placement, in order to shorten wire lengths. Here, any pin can be in any one out of several possible positions (for details, see [RW1]). By considering all possible positions of a pin to be a group of vertices, out of which just one needs to be connected, we are confronted with the following generalization of SP: For an undirected,
connected distance graph with groups of required vertices and optional Steiner vertices, compute a shortest connected subgraph of the input graph, containing at least one vertex of each group. We call this generalized version of SP the Group Steiner Problem (GSP). As the Steiner Problem is NP-hard, GSP is too, and we are interested in approximation algorithms. Some efficient ones have already been given by [RW1], but nothing about the quality of the approximate solutions is known so far. In practice, guarantees for approximate solutions are an important first step in learning about the tradeoffs between efficiency of the computation and the optimality of the solution.

In this paper we present polynomial time algorithms, where the quality of the computed approximate solution is bounded. In Section 2 we introduce some definitions and formulate precisely the optimization problems SP and GSP. Section 3 presents a simple polynomial time approximation algorithm for GSP, computing a tree that is no more than \( g - 1 \) times the length of a minimal tree, where \( g \) is the number of the required groups. We also prove \( g - 1 \) to be the sharpest bound for the proposed approximation algorithm. In Section 4, we will give an extended version of this algorithm, tightening the quality bound to \( g - h + 1 \), for an arbitrary positive integer \( 2 \leq h \leq g \). This increases the time complexity to a function exponential in \( h \). Moreover, we prove in Section 5 the quality bound \( g - 1 \) for a modification of an approximation algorithm proposed in [RW1] which is more efficient and computes better solutions in practice than the one given in Section 3.

2 GSP is a generalization of the Steiner Problem

We will consider undirected distance graphs only. A graph \( G = (V, E, l) \), where \( V \) is the set of vertices in \( G \), \( E \subseteq \{(v, w) : v, w \in V \land v \neq w\} \) is the set of edges in \( G \) and \( l : E \rightarrow [0, \infty) \) is a distance function. We call a graph \( G_1 \), a subgraph of \( G \), writing \( G_1 \subseteq G \), if \( V_1 \subseteq V \), \( E_1 \subseteq E \), and \( l_1 \) is a restriction of \( l \) to \( E_1 \). A path \((v, w)\) in \( G \) between vertices \( v \) and \( w \) where \( v, w \in V \) is a subgraph \( G_1 \) with \( V_1 = \{v, v_1, \ldots, v_k, w\} \) and \( E_1 = \{(v, v_1), (v_1, v_2), \ldots, (v_k, w)\} \). Writing \( e \in G \) or \( v \in G \) where \( e \) is an edge and \( v \) is a vertex means \( e \in E \) or \( v \in V \). The total length \( l(G) \) of a graph \( G \) is defined as the sum \( l(G) = \sum_{e \in E} l(e) \). If \( l(G) \leq l(G') \), then we say \( G \) is shorter than \( G' \). Consider \( \text{path}(v, v) \) as a graph \( \{(v), \emptyset, l\} \) with a total length equal to zero. We call a graph \( G \) connected if \( \forall v, w \in V \exists \text{path}(v, w) \subseteq G \). A graph \( G \) is a tree, if \( \forall v, w \in V \) there is exactly one \( \text{path}(v, w) \subseteq G \). A leaf of a tree \( T \) is a vertex \( v \in T \) with \( (v, w_1), (v, w_2) \in T \Rightarrow w_1 = w_2 \). A spanning tree \( T \) for a set of vertices \( \{v_1, \ldots, v_k\} \subseteq G \) of \( G \) is a tree where \( v_1, \ldots, v_k \in T \land T \subseteq G \). For brevity, a spanning tree of \( G \) denotes a spanning tree for all vertices of \( G \).

A connected graph \( G \) is a Steiner Graph (SG) if \( V = S \cup R \) and \( S \cap R = \emptyset \) with the set of Steiner vertices \( S \neq \emptyset \) and the set of required vertices \( R \neq \emptyset \). \( T \) is a Steiner Tree (ST) of \( G \) if \( T \) is a spanning tree for \( R \) of \( G \). The Steiner Problem (SP) asks for a minimal ST of \( G \), i.e., a ST of minimal total length among all ST of \( G \).

Our generalized version of SP is the Group Steiner Problem (GSP), introduced by [RW1]. A connected graph \( G \) is a Group Steiner Graph (GSG) if \( V = S \cup \bigcup_{i=1}^{g} R_i \) and \( g \) is a positive integer, \( \forall i, j \in \{1, \ldots, g\}, i \neq j : R_i \cap R_j = \emptyset \), and \( \forall i \in \{1, \ldots, g\} : (R_i \neq \emptyset \land S \cap R_i = \emptyset) \). We call the sets \( R_i \) groups of required vertices, and \( g \) the number of groups. Sometimes we will use the abbreviations \( n := |V|, m := |E| \) and \( n_i := |R_i| \). \( T \) is