ON THE RECTILINEAR ART GALLERY PROBLEM
- ALGORITHMIC ASPECTS -

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Abstract
We investigate the watchman problem for rectilinear art galleries with an arbitrary number of holes. An efficient algorithm for the placement of the guards with running time $O(n^{3/2} \log^2 n \log \log n)$ is presented. Each guard has to watch an $r$-star of constant size.

1. Introduction
How to watch a polygon ("art gallery") is a well-studied problem in Computational Geometry. The original Art Gallery Problem raised by V. Klee asks how many guards are sufficient to watch any $n$-sided polygon. For simple rectilinear polygons Kahn, Klawe, Kleitman [KKK] proved that $\lceil n/4 \rceil$ is the optimal bound. Later it has been shown that one can even decompose such polygons in $\lceil n/4 \rceil$ $r$-stars of size $\leq 6$, see the monograph [O'R] of J. O'Rourke, which contains almost all material on Art Gallery Problems. Recently, the first author [H] proved that the $\lceil n/4 \rceil$-bound also holds in the case of rectilinear polygons with an arbitrary number of holes if the guards are allowed to sit in any point of the gallery (point guards). This is the first art gallery theorem with a tight bound for general polygons. The aim of this paper here is to analyze the algorithmic aspects and the complexity of finding an $\lceil n/4 \rceil$-solution in the general case.

More precisely,
(1) we show that any $n$-sided rectilinear polygon can be decomposed into $\leq \lceil n/4 \rceil$ $r$-stars, each of size $\leq 16$. This improves [H] where the size of the stars was not bounded.
(2) we provide an $O(n^{3/2} \log^2 n \log \log n)$-time bounded algorithm to find such a solution.

Let us recall that Sack and Toussaint [ST] proved an $O(n \log \log n)$ bound in the simply connected case using triangulation. Moreover, we remark that the complexity (NP-completeness?) of finding a minimal solution for a given rectilinear polygon remains open, even in the case with holes.

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The paper is organized as follows. Section 2 introduces basic definitions and concepts. In Section 3 we outline the proof for the \([n/4]\)-bound and Section 4 provides the implementation and gives the complexity analysis of the algorithm.

2. Basic definitions

Let \(P\) denote a rectilinear polygon possibly with holes, i.e. all its sides are either horizontal or vertical. By \(\text{size}(P)\) we denote the number of its vertices and by \(h(P)\) the number of its holes. \(P\) is 1-connected ("simply connected") if \(h(P) = 0\). We say a polygon \(P'\) is smaller than \(P\) if \(h(P') < h(P)\), or if \(h(P') = h(P)\) and \(\text{size}(P') < \text{size}(P)\).

If \(x, y\) are points in \(P\), we denote by \([x, y]\) the closed line segment and by \((x, y)\) the open line segment. The point \(x\#y\) is that point whose horizontal coordinate is that of \(x\) and whose vertical coordinate is that of \(y\). \(R[x, y]\) is the rectangle spanned by the points \(x, y, x\#y, y\#x\). \(R(x, y)\) denotes the interior of this rectangle. The two coordinates of point \(x\) are denoted by \(x_1\) and \(x_2\).

It is possible that \(R[x, y] = [x, y]\). Two points \(x, y \in P\) are \(r\)-visible to one another in \(P\) if \(R[x, y] \subset P\). We write \(x \sim y\) in this situation. A line segment \(e = [y, y']\) is visible (sometimes called \(x\)-visible) if all points on the segment are \(r\)-visible to a given point \(x\). A \(x\)-visible corner \(y\) is totally visible from a point \(z\) if both neighbors of \(y\) are also \(x\)-visible. The \(v\)-region of a segment \(e\) is the maximal rectangle \(V(e)\), such that one side of \(V(e)\) is \(e\) and \(V(e) \subset P\). A line segment \(e = [x, x']\) is visible from another segment \(e' = [y, y']\) if the side of the \(v\)-region of \(e\) opposite to \(e\) is a subset of \(e'\). It is clear that \(r\)-visibility is a strictly stronger notion than usual visibility. Throughout this paper, we will mean \(r\)-visibility when we talk about visibility.

An \(r\)-star is a rectilinear polygon \(P\) such there is a point \(x \in P\) for which \(x \sim y\) for any \(y \in P\). All points \(x \in P\) with this property form the kernel \(\ker(P)\).

Let \(D = \{n := \text{north}, w := \text{west}, s := \text{south}, e := \text{east}\}\) denote the set of the main compass directions. If \(d \in D\) we define \(d^{-1}\) to be the inverse direction. Now we want to define the neighbor of some \(z \in P\) in direction \(d\), which we denote by \(d(z)\). If \(z\) is in the interior \(\text{int}(P)\) then \(d(z)\) is that point \(y\) on the boundary \(\text{bd}(P)\), which lies in direction \(d\) and \((x, y) \subset \text{int}(P)\). If \(z \in \text{bd}(P)\) then \(y = d(z)\) is that point in direction \(d\) on \(\text{bd}(P)\) for which either \([x, y] \subset \text{bd}(P)\) and \(y\) has maximal distance, or \((x, y) \subset \text{int}(P)\).

Further basic concepts are empty convex corners, \(X\)-shapes, \(L\)-shapes and \(T\)-shapes.

1. Let \(x\) be a convex corner with neighbors \(y\) and \(z\). We say \(x\) is empty, if \(R(y, z) \subset P\).
2. Let \(x\) be a convex corner with neighbors \(y, z\) and let \(x'\) be a concave corner with neighbors \(y', z'\) such that \(x' \in R(y, z)\). We say \(\{x, x'\}\) form an \(L\)-shape, if (i) \(x \sim z'\) and