Transfinite Reductions in Orthogonal Term Rewriting Systems

(Extended abstract)

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Abstract. Strongly convergent reduction is the fundamental notion of reduction in infinitary orthogonal term rewriting systems (OTRSs). For these we prove the Transfinite Parallel Moves Lemma and the Compressing Lemma. Strongness is necessary as shown by counterexamples. Normal forms, which we allow to be infinite, are unique, in contrast to \(\omega\)-normal forms. Strongly converging fair reductions result in normal forms.

In general OTRSs the infinite Church-Rosser Property fails for strongly converging reductions. However for Böhm reduction (as in Lambda Calculus, subterms without head normal forms may be replaced by \(\bot\)) the infinite Church-Rosser property does hold. The infinite Church-Rosser Property for non-unifiable OTRSs follows. The top-terminating OTRSs of Dershowitz c.s. are examples of non-unifiable OTRSs.

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1. INTRODUCTION

The theory of Orthogonal Term Rewrite Systems (OTRS) is now well established within theoretical computer science. Comprehensive surveys have appeared recently in [Der90a, Klo91]. In this paper we consider extensions of the established theory to cover infinite terms and infinite reductions.

1.1. Motivation

At first sight, the motivation for such extensions might appear of theoretical interest only, with little practical relevance. However, it turns out that both infinite terms and infinite rewriting sequences do have practical relevance.

A practical motivation for studying infinite terms and term rewriting arises in the context of lazy functional languages such as Miranda [Tur85] and Haskell [Hud88]. In such languages it is possible to work with infinite terms, such as the list of all Fibonacci numbers or the list of all primes. This style of programming has been advocated by Turner [Tur85], Peyton-Jones [Pey87] and others. Of course the outcome of a particular computation must be finite, but it is pleasant to define such results as finite portions of an infinite term. It would be even more pleasant to know that nice properties (for example Church-Rosserness) hold for infinite as well
as finite rewriting, but the standard theory does not tell us this. As we show below, Church-Rosserness is one of several standard results which does not hold for infinite rewriting in general, although it does hold for terms which have an infinite normal form (Theorem 4.1.3).

A second practical motivation for considering infinite reduction sequences arises from the common graph-rewrite based implementations of functional languages. The correspondence between graph rewriting and term rewriting was studied in [Bar87] for acyclic graphs. When cyclic graphs are considered, the correspondence with term rewriting immediately requires consideration of infinite terms and infinite reductions. The correspondence with graphs is the motivation for [Far89].

1.2. Overview

With these motivations in mind, we set out to identify precise foundations for transfinite rewriting. A certain amount of care is needed to establish appropriate notions and we do this in Section 2. One can take a topological approach as in [Der89a,b&90] and consider infinite reduction sequences that are converging to a limit in the metric completion of the space of finite terms. However, converging reductions fail to satisfy some natural properties for orthogonal TRSs. Instead we concentrate on strongly converging reductions as introduced by [Far89], which turn out to be better behaved.

<table>
<thead>
<tr>
<th>Basic facts for infinitary orthogonal term rewrite systems</th>
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<tbody>
<tr>
<td><strong>converging reductions</strong></td>
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<tr>
<td>Transf. Parallel Moves Lemma</td>
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<tr>
<td>Inf. Church-Rosser Property</td>
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<td>Unique $\omega$-normal forms</td>
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<td>Unique normal forms</td>
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<td>Compressing Lemma</td>
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<td>Fair reductions result in $\omega$-normal forms [Der90b], (3.4.2.i)</td>
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(Table 1.1)

In Section 3 we prove the fundamental results for infinitary orthogonal rewrite systems, as summarized in Table 1.1. Then in Section 4 we show the failure of the infinite Church-Rosser Property for general OTRSs. The counterexample refutes not only the CR-property for strongly converging but also the CR-property for converging reductions studied by Dershowitz c.s. Introducing ideas from Lambda Calculus we eliminate subterms that have no head normal form by reducing them to $\bot$. The new $\text{Böhm-reduction } \rightarrow_{\bot}$ has the infinite Church Rosser Property for strongly converging reductions. Normal forms for $\rightarrow_{\bot}$-reduction are so called Böhm Trees: they are unique. Finally we show that orthogonal TRSs in which there are no rule in which a left-hand side of a rule can be unified with the right-hand side have the infinite Church-Rosser Property. This class of orthogonal TRSs includes the top-terminating orthogonal TRSs of Dershowitz c.s.