Parallel Transport of Phases

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Abstract: General features of the concept of Berry’s phase are reported and extended to parallel transport based on curves of density operators. Product integral representations and a natural connection are introduced.

1 Introduction

Parallel transport of phases is a natural structure in the fundamentals of Quantum Theory. It is my aim to describe some essentials of that structure according to Berry [1] and Simon [2], which is defined via transport conditions for vectors and phases along curves of pure states. A further purpose is to introduce the extension of these constructions to curves of more general states (i.e. mixtures) [3]. To do so is a problem of internal consistency: In Quantum Theory - and in contrast to Classical Statistical Mechanics - the question whether a state is a pure or a mixed one is decided by the set of observables and can, consequently, be changed by adding or neglecting observables (operators). The criteria for parallelity should be compatible with this feature. On the other hand, the case of pure states is basic and most important, and serves as a guide. See also [4].

The vectors of a Hilbert space $\mathcal{H}$ represent pure states if two of them can be distinguished by their expectation values provided they are linearly independent. To do so one needs enough observables acting as operators on $\mathcal{H}$. The simplest and also natural assumption for this is that potentially every self-adjoint operator is allowed to become an observable. It is however sufficient, and for technical reasons highly desirable, to use the bounded hermitian operators on $\mathcal{H}$, i.e. the hermitian elements of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators acting on $\mathcal{H}$.

A vector $\psi$ describes a state by the collection of its expectation values

$$A \mapsto \frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle}$$

and for this reason two vectors describe the same state if and only if they are linearly dependent. Excluding the zero of $\mathcal{H}$ and identifying two linearly dependent
vectors defines the projective space, $\mathbb{P}\mathcal{H}$, which labels uniquely the pure states. It can hence be considered as the space of pure states. $\mathbb{P}\mathcal{H}$ can be realized either

a) as the space of 1-dimensional linear subspaces of $\mathcal{H}$ – the first Grassmann manifold of $\mathcal{H}$, or
b) as the space of rays of $\mathcal{H}$ (a ray is 1-dimensional linear subspace with the exclusion of the zero of $\mathcal{H}$), or
c) as the space of the 1-dimensional projection operators, i.e. of the operators $P = P^2 = P^*$ which project $\mathcal{H}$ onto an 1-dimensional subspace.

Here always exclusively $\mathbb{P}\mathcal{H}$ is interpreted as the set of 1-dimensional projections. The merit in doing so is: The points of $\mathbb{P}\mathcal{H}$ appear as operators, and $\mathbb{P}\mathcal{H}$ is canonically imbedded into $\mathcal{B}(\mathcal{H})$ as a subset. As an example, the distance between two elements of $\mathbb{P}\mathcal{H}$ can be given by the operator norm $\|P_2 - P_1\|$ of their difference. An inconvenience in using case c) above is in the double role the projections of rank one are playing: Such an operator represents as well a state as a genuine observable asking with which apriori probability this state is realized.

$\mathcal{H} - \{0\}$, the Hilbert space without its zero element, can be considered as a $\mathbb{C}^\times$-fibre bundle over $\mathbb{P}\mathcal{H}$. Because the norming of vectors is a topological trivial operation it is further useful to introduce the unit sphere

$$S(\mathcal{H}) = \{\psi \in \mathcal{H} : <\psi, \psi> = 1\} \tag{1}$$

of $\mathcal{H}$ which is a $S^1$-bundle over $\mathbb{P}\mathcal{H}$.

Every Schrödinger equation

$$H(t)\psi = i\dot{\psi} \tag{2}$$

determines a (non-canonical) lift from $\mathbb{P}\mathcal{H}$ for the integral curves of

$$[H(t), P] = i\dot{P}. \tag{3}$$

Indeed, if $t \mapsto P_t$ is a solution of (3) and $P_0 = |\psi_0 > < \psi_0|$ then there is just one solution $t \mapsto \psi(t)$ of (2) with $\psi(0) = \psi_0$. Now $t \mapsto \psi(t)$ is clearly a lift of $t \mapsto P_t$ into $\mathcal{H} - \{0\}$. This lift sits in the subbundle (1) because of the conservation of the norm.

Replacing within (2)

$$H(t) \mapsto H_{\text{new}}(t) = H(t) - a(t)I,$$

the new curve

$$H_{\text{new}}(t)\psi_{\text{new}} = i\dot{\psi}_{\text{new}}, \quad \psi_{\text{new}}(0) = \psi_0$$
in $\mathcal{H} - \{0\}$ is again a lift of $t \mapsto P_t$ with

$$\psi_{\text{new}}(t) = \exp i \int_0^t a(t)dt \cdot \psi(t).$$

This shows that the lifting may produce rather arbitrary phases. Furthermore, (2) and (3) produce lifts only for solutions, a rather restricted class of curves in the