On the complexity of algebraic power series

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1 Introduction

A classical computational model do deal with "computable" power series consists in giving an algorithm to compute their coefficients and to consider suitable truncations in order to perform the required operations. In the case of algebraic power series, i.e. when the series $f(X)$ is given by a polynomial $G(X_1,\ldots,X_n,T)$ s.t. $G(X_1,\ldots,X_n,f(X)) = 0$, these coefficients can be computed for instance using [K-T]. Since there is in general more than one series vanishing at the origin and satisfying the above identity, one must compute enough terms of the Taylor expansion of $f$, which, at least, permit to distinguish it from the other roots of $G$. It is clear that computational problems arise naturally in case the series $f$ should be used for further calculations, e.g. to determine the solution $h$ of a polynomial depending on the $X$ variables and on $f$.

In order to avoid these problems, in [AMI], we introduced a purely symbolic model of computation, based on the notion of Locally Smooth Systems (LSS), and we showed that these systems have good computational properties: standard bases and normal forms can be calculated in the ring of algebraic power series and it is possible to give effective versions of classical theorems like the Weierstrass Preparation Theorem and the Noether Normalization Lemma.

The aim of this paper is to look for suitable measures for the complexity of algebraic power series. In [R1], [R2] R.Ramanakoraisina defines the complexity of an analytic function $f$ satisfying a polynomial equation to be the degree $c(f)$ of its minimal polynomial; and he shows that this definition satisfies all the required properties of complexities. Since we are interested in the local properties of $f$ (at the origin) we consider here a notion which takes care of

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both the degree and the multiplicity defined as the minimum order \( e(f) \) of defining polynomials.

We introduce then a notion of complexity for algebraic series represented in model of computation of LSS’s. To do this we introduce suitable costs for Locally Smooth Systems by means of the length \( \lambda(f) \) (i.e. the number of extra variables) and the degree \( \delta(f) \) (i.e. the product of the degrees of the involved polynomials). We define then the complexity of \( f : \xi(f) = (\lambda(f),\delta(f)) \) as the minimum of the costs of all Locally Smooth Systems representing \( f \).

The main result of this note shows how these costs can be estimated in terms of degree and multiplicity : \( \lambda(f) \leq e(f) \) and \( \delta(f) \leq c(f)^{\lambda(f)} \leq c(f)^{e(f)} \). Conversely, we have that \( o(f) \leq c(f) \leq \delta(f) \), where \( o(f) \) is the order of \( f \) at the origin.

Then we introduce the complexity \( \xi' \) as the minimum cost of Standard Locally Smooth Systems defining \( f \), and we find bounds for \( \xi' \) in terms of \( \xi \) and of the maximum degree of involved polynomials. Using this we show that the cost of representing the Weierstrass form of a distinguished polynomial of order \( b \) can be estimated by \( (b(r + 1), D^b(r+1)) \) where \( r \) is the length of the involved Locally Smooth System and with \( D \) depending on the degrees of the data. Finally, using the above estimates, we find a test to check whether an algebraic function is indeed a rational function.

## 2 Notation and preliminaries

Let \( K \) be a subfield of the field of complex numbers, let \( X = (X_1, \ldots, X_n) \) a set of variables and \( K[[X]]_{alg} \) the algebraic closure of \( K[X] \) in \( K[[X]] \), which is the set of algebraic formal power series, and which is also the henselization of the ring of polynomials with respect to the maximal ideal corresponding to the origin.

Let us recall from [AMR] the notion of Locally Smooth Systems. We say that a system of polynomials \( F = (F_1, \ldots, F_r) \) is a LSS if the \( F_i \)'s are polynomials in \( K[X_1, \ldots, X_n, Y_1, \ldots, Y_r] \) vanishing at the origin and s.t. the Jacobian of the \( F_i \)'s with respect to the \( Y_j \)'s at the origin is a lower triangular non singular matrix, i.e. we can write:

\[
F_i(X, Y_1, \ldots, Y_r) = \sum_{j=1}^{r} c_{ij} Y_j + H_i(X, Y_1, \ldots, Y_r)
\]

with \( H_i \in (X, (Y_1, \ldots, Y_r)^2) \) and \((c_{ij})\) a non-singular lower triangular \( r \) by \( r \) matrix. Under this assumption, by the Implicit Function Theorem, there are unique algebraic series \( f_1, \ldots, f_r \in K[[X]]_{alg} \) s.t. \( f_j(0) = 0 \ \forall \ j \), and \( F_i(X, f_1, \ldots, f_r) = 0 \ \forall \ i \). We will also say that \( F = (F_1, \ldots, F_r) \) is a LSS for the \( f_i \)'s (or defining the \( f_i \)'s; or that the \( f_i \)'s are given by the LSS \( F \) etc.).

The key point of our approach in [AMR] was to look for results in \( K[[X]]_{alg} \) by working with suitable, and computable, extensions of \( K[X] \). Namely, given