Once More on Order-Sorted Algebras

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Abstract A new definition of order-sorted algebras is given which unifies the approaches to be found in the literature. We claim that some conceptual clarification is obtained without losing out on technical simplicity. The corresponding theory is developed and related to the literature. Specifically, we prove completeness with regard to an appropriate deduction system.

0. Order-sorted algebras have a long genesis. Since its first manifestation in computer science [Goguen 78], a variety of interpretations of "order-sorting" have surfaced (such as [Gogolla 84], [Poigné 84], [Smolka 86], [Poigné 88]) which reflect different assumptions. Some ideas are shared, for instance that subsorting denotes inclusion and that overloading is allowed.

Opinions are split whether to accept "ad-hoc polymorphism" [Strachey 67] as a semantic concept. Ad-hoc polymorphism refers to an accidental overloading of operators as often found in many-sorted signatures, where the same symbol is used for semantically unrelated operators, e.g. $+$: nat nat $\rightarrow$ nat and $+$: bool bool $\rightarrow$ bool. Goguen and Meseguer (e.g. [Goguen, Meseguer 89], up to now the last in a series of papers by the same authors) argue in favour of ad-hoc polymorphism, observing that otherwise many-sorted algebra is not subsumed. Other authors [Gogolla 84], Poigné 84] commit themselves to the slogan of "sort-independent semantics", meaning that an operation applied to data should always give the same result whatever typing is involved.

This short note aims at a reconciliation of both schools of thought handling ad-hoc polymorphism, but retaining the technical simplicity of the sort-independent semantics.

I acknowledge inspiration by Goguen's and Meseguer's paper [Goguen, Meseguer 89], though the basic idea is already to be found in [Poigné 88] in another formal framework. This paper being a study of foundations, we rely on the references for motivation and applications of order-sorted algebra.

1. Let us recall the definition of a many-sorted signature and of many-sorted algebras.

1.1 Definition • A many-sorted signature $(S, \Sigma)$ consists of a set $S$ (of sorts) and $S \times S$-sorted family $(\Sigma_w,s | w \in S^*, s \in S)$. Elements $\sigma \in \Sigma_{w,s}$ are called operation symbols, for short operators. We often use the notation $\sigma_{w,s} \in \Sigma$ instead of $\sigma \in \Sigma_{w,s}$.

• A many-sorted $(S, \Sigma)$-algebra $A$ consists of a set $s^A$ for every sort $s \in S$, and an operation (function) $\sigma_{w,s}^A : w^A \rightarrow s^A$ for every operator $\sigma_{w,s} \in \Sigma$ where $\lambda^A = \{ \varnothing \}$ and $w^A = w^S \times s^A$.

• A $(S, \Sigma)$-homomorphism $h : A \rightarrow B$ consists of a $S$-sorted family ($h_s : s^A \rightarrow s^B \; \forall s \in S$) of functions such that $h_s(\sigma_{w,s}^A(a_1,\ldots,a_n)) = \sigma_{w,s}^B(h_s(a_1),\ldots,h_s(a_n))$.

These data form a category $(S, \Sigma)$-Alg.

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NOTATION We use $a$ as abbreviation for tuples $a_1, \ldots, a_n$. The notation $a_i$ refers to the $i$'s component of a tuple. Functions, relations etc. are extended canonically to tuples, e.g. $a = b$ stands for $a_i = b_i$ for all $i = 1, \ldots, n$.

Order-sorting adds a partial order on sorts and allows for overloading. We typically find that an order-sorted signature consists of a triple $(S, \leq, \Sigma)$ such that $(S, \Sigma)$ is a many-sorted signature, and $(S, \leq)$ is a partially ordered set (poset). The subset relation $s \leq s'$ is semantically interpreted as inclusion $s^a \subseteq s'^a$ abandoning many-sortedness in that overloading cannot be eliminated by sort information any more.

Goguen and Meseguer recover many-sortedness by introduction of the monotonicity condition

$$\sigma \in \Sigma_{w,s} \cap \Sigma_{w',s'}, \ w \leq w' \implies s \leq s'$$

on signatures (where $w = s_1 \ldots s_n \leq s'_1 \ldots s'_n = w'$ if $s_i \leq s'_i$ for $i = 1, \ldots, n$), and by the condition

$$\sigma \in \Sigma_{w,s} \cap \Sigma_{w',s'}, \ w \leq w', \ a \in w^a \implies \sigma_{w,s}^a(a) = \sigma_{w',s'}^a(a).$$

as a semantical counterpart. Homomorphisms are restricted by

$$s \leq s', \ a \in s^a \implies h_s(a) = h_{s'}(a).$$

(We refer to such structures as GM-signatures, SIG-GM-algebras and SIG-GM-homomorphisms).

Overloading is ad-hoc since the operators with the same name may behave differently semantically. Specifically, if the order is discrete, many-sorted signatures and many-sorted algebras are obtained as a special case.

However, we notice a disturbing property: consider the signature with sorts $s_1 \leq s_3$, $s_2 \leq s_4$, $s_5$ and operators $a_1, b_2, \sigma_{s_1 s_4}, \sigma_{s_2 s_5}, \sigma_{s_3 s_2}, \sigma_{s_3 s_5}$. Then an algebra $A$ is well defined where $s_1^a = s_2^a = s_3^a = s_4^a = \{0\}$ and $s_5^a = \{1, 2\}$ and $\sigma_{s_1 s_4}^a(0, 0) = 1$, $\sigma_{s_2 s_5}^a(0, 0) = 2$. But the standard term construction $(\sigma(t)) \in t_x$, whenever $\sigma \in \Sigma_{w,s}$, $s \leq s'$ and $t \in t_w$, see 3.1) generates $\sigma(a, b)$ as only element of sort $s_3$, hence does not define an initial algebra. The problem disappears if the signature is regular, i.e. given $\sigma \in \Sigma_{w,s}$ and given $w \leq w''$ in $S^*$ there is a least rank $(w', s') \in S^* \times S$ such that $w \leq w'$ and $\sigma \in S_{w',s'}$.

Gogolla [Gogolla 83] as well as [Poigné 84, 90] (and similarly [Smołka 86]) impose the following semantical condition

$$\sigma \in \Sigma_{w,s} \cap \Sigma_{w',s'}, \ a \in w^a \cap w'^a \implies \sigma_{w,s}^a(a) = \sigma_{w',s'}^a(a).$$

Here sorts should be thought of as to determine subsets of a "universal" set $A = \bigcup_{s \in S} A_s$, and operators should be thought of as (totally defined) components of a partial function $\sigma^a: A^a \to A$. Data cannot be distinguished by sort information, hence the definition is (somewhat confusingly) one-sorted in spirit, the subsorts being only a means to introduce a certain degree of partiality.

In this paper, we propose to extend the latter approach to a many-sorted one: we consider a set $\{(S_s, \leq) | s \in S\}$ of partial orders instead of only one, and recast all definitions appropriately. The "sorts" $s \in S_s$ may be considered as subsorts of the principal sort $s \in S$ (for sake of a better name), the carrier $s^a$ being the union $\bigcup_{s \in S} A_s$. Operators $\sigma_{w,s}$ are instances of a global operator $\sigma$ on the respective principal sorts. The actual formalization is more or less to be a matter of taste. We propose the following definition which allows to rephrase the arguments of [Poigné 90] at little expense.

1.2 DEFINITION An order-sorted signature $(S, \leq, S, \Sigma)$ consists of a many-sorted signature $(S, \Sigma)$, a partial order $(S, \leq)$, and a partition $^1 S$ of $S$ such that

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1 A family $\{C_i \subseteq X | i \in I\}$ is called a partition of $X$ if $X = \bigcup_{i \in I} C_i$ and if $C_i \cap C_j = \emptyset$ if $i \neq j$ for all $i, j \in I$.\]