Towards a Theory of Wave Transport

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Abstract: A technique is described that should prove invaluable for the computation of effective diffusion coefficients due to wave motion.

1 Introduction

The purpose of this note is to describe a technique available for studying transport of contaminants (e.g. element abundances) by various types of wave motion. Recent progress in this area is discussed as are several relevant experiments. The article focuses on transport by simple waves; the origin of these waves is not addressed, nor is their generation. Some aspects of these topics are discussed by Knobloch (1990) with particular emphasis on the nonlinear processes that select among different possible waveforms. The waves are taken to be spatially periodic and deterministic; random waves are not considered. Particular emphasis is placed on the role played by microscopic diffusion.

2 General technique for calculating $\kappa^*$

The technique described below is called the homogenization method and in the form described here is due to Papanicolaou and Pirroneau (1981, see also McLaughlin et al 1985). Consider the advection-diffusion equation

$$\frac{\partial C}{\partial t} + u \cdot \nabla C = \kappa \nabla^2 C,$$

where $C \equiv C(x, t)$, subject to the initial condition $C(x, 0) = C_0(x)$, say. The prescribed velocity field $u(x, t)$ is assumed to be incompressible and periodic in both $x$ and $t$ with zero mean. In the following the notation $< ... >$ is used for spatial averages, while an overbar will be used to denote time averages. Suppose that $C(x, 0)$ is slowly varying, so that

$$C(x, 0) = C_0(\varepsilon x), \quad 0 < \varepsilon \ll 1,$$

where $\varepsilon = \ell/L$, with $\ell$ the length scale of $u(x, t)$ and $L$ the length scale of $C(x, 0)$. Equation (2) suggests multiple scale analysis, with

$$X = \varepsilon x, \quad T = \varepsilon^2 t$$
as the slow variables. The solution of (1) then takes the form
\[ C(x, t; \epsilon) = C_0(X, T) + \epsilon C_1(x, t; X, T) + \epsilon^2 C_2(x, t; X, T) + \ldots, \] (4)
where the \( C_j (j \geq 1) \) are periodic in \( x, t \). The quantity of interest is the large scale field \( C_0(X, T) \). Its evolution is obtained by usual asymptotic methods. The \( O(\epsilon^0) \) equation is satisfied identically; at \( O(\epsilon) \) one obtains an evolution equation for \( C_1 \),
\[ \frac{\partial C_1}{\partial t} + u \cdot \nabla_x C_1 - \kappa \nabla_x^2 C_1 = -u \cdot \nabla_x C_0, \] (5)
whose solution takes the form \( C_1 = v \cdot \nabla_x C_0 \), with \( v(x, t) \) satisfying the equation
\[ \frac{\partial v}{\partial t} + u \cdot \nabla_x v - \kappa \nabla_x^2 v = -u. \] (6)
At \( O(\epsilon^2) \) one obtains
\[ \frac{\partial C_2}{\partial t} + \frac{\partial C_0}{\partial T} + u \cdot \nabla_x C_2 + u \cdot \nabla_x C_1 = \kappa \nabla_x^2 C_0 + 2\kappa \nabla_x \cdot \nabla_x C_1 + \kappa \nabla_x^2 C_2. \] (7)
This equation has a solution \( C_2 \) that is periodic in \( x, t \) if and only if the following solvability condition holds:
\[ \frac{\partial C_0}{\partial T} = \kappa \nabla_x^2 C_0 - \frac{1}{2} \left( \langle \nabla_x v_{ij} \rangle + \langle \nabla_x v_{ji} \rangle \right) \equiv \kappa_{ij}^* \frac{\partial^2 C_0}{\partial X_i \partial X_j}, \] (8)
where
\[ \kappa_{ij}^* = \kappa \delta_{ij} - \frac{1}{2} \left( \langle v_{ij} \rangle + \langle v_{ji} \rangle \right) \] (9)
is the effective Eulerian diffusivity. Given a velocity field \( u(x, t) \), we can solve (6) for \( v \) and hence find \( \kappa_{ij}^* \).

An alternative form of \( \kappa_{ij}^* \), obtained from (9) by simple manipulation, is sometimes useful:
\[ \kappa_{ij}^* = \kappa \{ \delta_{ij} + \langle \partial_k v_i \partial_k v_j \rangle \}, \] (10)
and shows that the diffusivity enhancement \( \kappa_{ij}^* - \kappa \delta_{ij} \) is a positive definite tensor. This expression does not imply, however, that \( \kappa_{ij}^* \) vanishes when \( \kappa = 0 \). In this case equation (6) has the exact Lagrangian solution (e.g., Frisch 1988)
\[ v^L(a, t) = -\int_0^t u^L(a, t')dt' \] (11)
with \( a \) denoting the initial condition \( x = a \) at \( t = 0 \). Then
\[ \kappa_{ij}^* = \frac{1}{2} \int_0^t \left[ \langle u_i^L(a, t) u_j^L(a, t') \rangle + i \leftrightarrow j \right] dt' \] (12)
which is just Taylor’s diffusivity. The conditions under which this integral converges, and hence defines a true diffusivity, are still not fully understood. In general some microscopic diffusion is necessary to render the advection of a passive contaminant truly diffusive.