ABOUT THE EFFECT OF THE NUMBER OF SUCCESSFUL PATHS IN AN INFINITE TREE ON THE RECOGNIZABILITY BY A FINITE AUTOMATON WITH BUCHI CONDITIONS

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Abstract.

We modify an acceptance condition of Büchi automaton on infinite trees: rather than to require that each computation path is successful, we impose various restrictions on the number of successful paths in a run of the automaton on a tree. All these modifications alter the recognizing power of Büchi automata. We examine the classes induced by the acceptance conditions that require \(_{<=}\alpha, \_\geq\alpha, \_\leq\alpha\) successful paths, where \(\alpha\) is a cardinal number. It turns out that, except some trivial cases, the "\(_{<=}\)" classes are incomparable with the class \(Bü\) of Büchi acceptable tree languages, while the classes "\(_{\geq}\)" are strictly included in \(Bü\).

Introduction

Automata on infinite sequences and infinite trees, introduced by Büchi (1962) [1] and Rabin (1969) [3] in context of decision procedures for monadic arithmetics of one or many successors respectively, have been more recently recognized as a tool in those areas of computer science where infinite computations are meaningful (see Thomas (1990) [6] for a survey).

One major field of applications consists of elementary decision procedures for a variety of propositional logics of programs, including propositional dynamic logic and its extensions, \(\mu\)-calculus and branching time temporal logic. The idea is, given a logical formula \(\phi\), to construct an automaton \(A_\phi\) accepting exactly the tree models of \(\phi\): then a decision procedure for satisfiability problem follows from that for emptiness problem, the latter depending on the type of automata actually used. In this context, the class of Büchi automata on infinite trees that is a proper subclass of Rabin automata, for which the emptiness problem is apparently more feasible (P vs. \(NP\) cf. Vardi and Wolper (1984)[8], Emerson and Julta (1988)[2]) seems to form an intriguing borderline, both in complexity and expressiveness: some of the logics being reducible to Büchi automata, the others require the full strength of Rabin automata which makes them more difficult to decide (cf. Vardi and Stockmeyer (1985)[7]). Several other variants of automata on infinite trees have been also proposed, usually motivated by the logics in consideration (Street (1982)[5], Vardi and Stockmeyer (1985)[7]).

In the present paper, we propose a modification of Büchi automaton based on a change of the acceptance condition.

The classical paradigm of an accepting run of an automaton on a tree requires that a success is achieved on each path of the computation. We relax this postulate and consider other, more flexible conditions that can be imposed on the set of successful paths. For example, one can be interested in runs, where the set of successful paths is infinite, or uncountable, or dense. In a general setting, for a set of cardinals \(\Gamma\), we define the set \(L_\Gamma(A)\) of trees accepted by an automaton \(A\) in the sense that there exists a run of successful paths of which belongs to \(\Gamma\). This induces the classes \(r-Bü\), in particular \((\leq\alpha)-Bü, (\geq\alpha)-Bü, (\leq\alpha)-Bü\), for a cardinal number \(\alpha\), that we want to relate to the class \(Bü\) of tree languages accepted by (classical) Büchi automata. It turns out that all the modifications in consideration

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alter the recognizing power of Büchi automata: there exists a Büchi recognizable set of trees that is not in any $\mathcal{R}$. On the other hand, one can observe a contrast between "negative" conditions $\leq \alpha$, $\prec \alpha$ and "positive" conditions $\geq \alpha$, $\succ \alpha$: except for some trivial cases, the classes $(\leq \alpha)-\mathcal{B}_{\mathcal{R}}$, $(\prec \alpha)-\mathcal{B}_{\mathcal{R}}$ are incomparable to $\mathcal{B}_{\mathcal{R}}$, while the classes $(\geq \alpha)-\mathcal{B}_{\mathcal{R}}$, $(\succ \alpha)-\mathcal{B}_{\mathcal{R}}$ are strictly included in $\mathcal{B}_{\mathcal{R}}$ (for $(\leq \alpha)-\mathcal{B}_{\mathcal{R}}$, the answer depends on $\alpha$).

The paper is organized as follows. After giving the preliminaries in Section I, we present, in Section II, the examples that prove the non-inclusions between the classes in consideration. Section III is devoted to the inclusion results mentioned above, we also prove that the families $(\leq \alpha)-\mathcal{B}_{\mathcal{R}}$, $(\prec \alpha)-\mathcal{B}_{\mathcal{R}}$, $(\leq \alpha)-\mathcal{B}_{\mathcal{R}}$ are included in the class of Rabin recognizable tree languages. In Section IV we briefly discuss a "density" condition for successful paths and, in the last section, outline the directions for further research.

I. Preliminaries

We denote by $\{0, 1\}^*$, (resp. $\{0, 1\}^\omega$) the set of finite (resp. infinite) words on the alphabet $\{0, 1\}$. Let $f$ be a mapping: $A \rightarrow B$ and $A' \subseteq A$; we denote by $\text{Inf}(f \upharpoonright A')$ the set $\{y \in B/ f^{-1}(y) \cap A' \geq \omega \}$. Given a finite alphabet $\Sigma$, a $\Sigma$-tree $t$ is a mapping: $D_t \rightarrow \Sigma$, where $D_t$ is a prefix-closed subset of $\{0, 1\}^*$, called the domain of $t$. The empty word denoted by $e$ is the root of the tree. A tree is finite, resp. infinite if its domain is finite, resp. infinite. An element $x$ of $D_t$ is a node of the tree, and its label is $t(x)$. Given a tree $t$ with domain $D_t$, a path of $t$ is a maximal word of $D_t$ or an infinite word $p$ of $\{0, 1\}^\omega$ such that every left factor of $p$ belongs to $D_t$. We denote by $T_\Sigma$ the set of finite $\Sigma$-trees and $T_\Sigma^\omega$ the set of all the $\Sigma$-trees. To define operations on trees we need to introduce trees with variables. Let $V$ be a finite set of variables (disjointed of $\Sigma$). A tree with variables is obtained by replacing subtrees by single nodes labelled with variables. Let $T_\Sigma(V)$ and $T_\Sigma^\omega(V)$ be the corresponding sets of trees with variables.

Let $t, t' \in T_\Sigma(V)$, $x \in D_t$, and $v \in V$. We will use the three following operations on $T_\Sigma(V)$:

- $t_x$ is the subtree of $t$ with root $x$, its domain is $x^{-1}D_t$.
- $t(x \leftarrow v)$ is the tree obtained by replacing in $t$ the subtree with root $x$ by a single node with label $v$.
- Actually, this operation is iterated on elements of $T_\Sigma$ to obtain the set $T_\Sigma(V)$.

We shall also use an extension of this last operation to a sequence of variables instead of a single one. Let $T_1, T_2, \ldots, T_m \subseteq T_\Sigma(V)$, and $c = (v_1, \ldots, v_m)$, we denote by $T.\overline{c}(T_1, \ldots, T_m)$ the set of trees obtained by substituting each occurrence of $v_i$ by some tree in $T_i$. Moreover, we define the set $T.\overline{c}(T_1, \ldots, T_m)^\omega$ as the set of all the infinite trees obtained by $\omega$-fold iteration of this $c$-concatenation.

A tree language is a subset of $T_\Sigma$.

Tree-automata.

A (nondeterministic top-down) Büchi tree automaton over $\Sigma$ is a 4-uple $A = (Q, \lambda, Q_0, F)$, where $Q$ is a nonempty finite set of states, $Q_0, F \subseteq Q$ are respectively the set of initial and final states and $\lambda \subseteq Q \times \Sigma \times Q \times Q$ is the transition relation. A run of $A$ on a tree $t \in T_\Sigma$ is an infinite $Q$-tree $r$ such that $r(e) \in Q_0$, $(r(x), t(x), r(x_0), r(x_1)) \in \lambda$ for every $x \in \{0, 1\}^*$. A tree $t$ is accepted by $A$ (according to Büchi condition) if there exists a successful run $r$ of $A$ on $t$, that means a run such that every infinite path $\pi$ is successful in the following sense: $\inf(r | \pi) \cap F \neq \emptyset$. A language $T \subseteq T_\Sigma$ is Büchi-recognizable if there is an automaton $A$ such that $T$ is the set of trees accepted by $A$. The language accepted by $A$ is denoted by $L(A)$. We denote by $\mathcal{B}_{\mathcal{R}}$ the family of Büchi-recognizable languages over $\Sigma$.

There exists a simple representation of Büchi-recognizable languages in terms of recognizable sets of finite trees given by Rabin[]:

Theorem I.1. — A language $T \subseteq T_\Sigma$ is Büchi-recognizable iff there are recognizable languages $T_0, T_1, \ldots, T_m \subseteq T_{\Sigma^\omega}$, where $V = (v_1, \ldots, v_m)$, such that $T = T_0.\overline{c}(T_1, \ldots, T_m)^\omega$. Our purpose is to modify this acceptance condition by changing the number of successful paths; in the Büchi condition all the paths have to be successful ones. Let $\alpha$ be the cardinality of the set of successful paths in a run of the automaton over a tree. We have $\alpha \leq 2^\omega$. We will denote respectively by $L_{\leq \alpha}(A)$, $L_{\prec \alpha}(A)$, $L_{\geq \alpha}(A)$, $L_{\succ \alpha}(A)$, $(\leq \alpha)-\mathcal{B}_{\mathcal{R}}, (\prec \alpha)-\mathcal{B}_{\mathcal{R}}, (\geq \alpha)-\mathcal{B}_{\mathcal{R}}, (\succ \alpha)-\mathcal{B}_{\mathcal{R}}$ the language (the family of languages) accepted by an automaton $A$ where the acceptance condition is: there