CUTTING PLANE VERSUS FREGE PROOF SYSTEMS

Andreas Goerdt
Universität-GH-Duisburg
Fachbereich Mathematik/Praktische Informatik
Lotharstraße 65, D-4100 Duisburg
Germany
Net address: hn281go@unidui.uucp

ABSTRACT
The cutting plane proof system for proving the unsatisfiability of propositional formulas in conjunctive normal form is based on a natural representation of formulas as systems of integer inequalities. We show: Frege proof systems p-simulate the cutting plane proof system. This strengthens a result in [5], that extended Frege proof systems (which are believed to be stronger than Frege proof systems) p-simulate the cutting plane proof system. Our proof is based on the techniques introduced in [2].

INTRODUCTION
The systematic study of the complexity of proof length in various proof systems for propositional logic (like resolution [3], tableaux calculus [11], Frege systems (or Hilbert style systems, these are the usual systems with modus ponens)) was initiated by Cook/Reckhow in [6]. One motivation for this is the following: As propositional proof systems are just nondeterministic algorithms for the coNP-complete language of unsatisfiable propositional formulas (or equivalently tautologies) the NP ≠ coNP assumption implies the existence of hard examples for any proof system. Hard examples are infinite families of formulas having only proofs of superpolynomial size in the system considered. It is an interesting problem, to show the existence of hard examples for increasingly powerful proof systems, without assuming NP ≠ coNP. Even for the relatively weak resolution proof system the existence of hard examples was proved by Haken only in 1985 [7]. Ajtai [1] showed this result for bounded depth Frege systems (which are stronger than resolution). Both authors use a family of formulas encoding the pigeonhole principle (saying that a total mapping from a set with n+1 elements to a set with n elements is not injective). The existence of hard examples for Frege systems with unbounded formula depth (or for stronger systems) is a well-known open problem [4]. Buss [2] proved that Frege systems allow for short (i.e. polynomial) proofs of the pigeonhole principle.

The cutting plane proof systems is based on the representation of propositional formulas in conjunctive normal form as systems of integer inequalities (see example 1.3 below). The only paper investigating the complexity of the cutting plane system is [5], where the following results are shown: The cutting plane proof system p-simulates (see definition 1.1 below) resolution. The cutting plane proof system has short
proofs for the formulas encoding the pigeonhole principle. Extended Frege system p-simulate the cutting plane system. The existence of hard examples is mentioned as an open problem. Our result that (not extended) Frege systems p-simulate the cutting plane system shows that finding hard examples for Frege systems is at least as difficult as finding hard examples for the cutting plane system. So perhaps one should try to find hard examples for the cutting plane system before looking at Frege systems. An application of our result is the following: It is possible to prove that a family of formulas has short Frege proofs by constructing short cutting plane proofs. As the cutting plane system is in some respects easier to handle than Frege systems, cutting plane proofs can be simple where Frege proofs are complex. We demonstrate this for a family of formulas introduced in [9], not known to have short Frege proofs before. (Here also [10] might be of interest.)

The second motivation of our paper comes from the fact, that propositional proof systems are the basis of algorithms for the satisfiability problem. In particular, work has been done on applying techniques from linear integer programming (like cutting planes) to obtain algorithms for the satisfiability problem (see for example [8]). Here we contribute to an analysis of the principles underlying these algorithms and to a comparison of these algorithms to the more common logic based ones.

To prove our result, we have to simulate the parts of arithmetic available in the cutting plane system in Frege systems. For this we use the techniques introduced in [2]. As the situation here is more general than in [2], we have to simulate more of arithmetic (multiplication, negative numbers, subtraction) in Frege systems.

In section 1 we start with some basics and give an example of a cutting plane derivation. In sections 2 and 3 we present our simulation of cutting plane proofs by Frege proofs. Finally we give the application of our result. Due to space restrictions we have omitted some proofs. In [12] you can find an appendix of this paper containing all proofs missing here.

1 BASICS

For undefined notions of propositional logic we refer the reader to § 2 of [2]. In particular, by Frege proof system we mean the Frege system presented there. By the size of a proof in any propositional proof system we mean the length of the proof written out as a string over a binary alphabet. Note that the size of a proof also accounts for the size of the formulas occurring in the proof whereas the number of proof steps of a proof does not.

1.1 Definition ([6], definition 1.5)

Let \( P \) and \( Q \) be two propositional proof systems. System \( P \) p-simulates system \( Q \) iff there is a polynomial \( p(x) \) such that for any proof \( Q \) of the formula \( F \) in system \( Q \) there is a proof \( P \) of \( F \) (or a formula corresponding to \( F \)) in \( P \) such that size \( P \leq p(\text{size } Q) \).