Abstract: We shall show that the finite Ramsey theorem as a $\Delta_0$ schema is provable in $\text{IA}_0^{+\Omega_1}$. As a consequence we get that propositional formulas expressing the finite Ramsey theorem have polynomial-size bounded-depth Frege proofs.

1 Introduction

In fragments of arithmetic such as $\text{IA}_0$ and $\text{IA}_0^{+\Omega_1}$ the exponentiation is only a partial function. Many combinatorial and number theoretical statements, though they are $\Pi_1$, need for their proofs the $\Pi_2$ axiom $\text{Exp}$ saying that exponentiation is a total function. This leads to the question what $\Pi_1$ principles are needed to derive $\Pi_1$ consequences of exponentiation over $\text{IA}_0$ or $\text{IA}_0^{+\Omega_1}$ as a base theory. Let us consider $\Delta_0$-$\text{PHP}$, the $\Delta_0$ Pigeon Hole Principle, as an example. This is the schema, for every $\Delta_0$ formula $\varphi$, $\forall u \left( \forall x \leq u \exists y < u \varphi(x,y) \rightarrow \exists x_0 < x \leq u \exists y < u \varphi(x_0,y) \& \varphi(x_1,y) \right)$. The meaning of the schema is that no mapping from $[0,u]$ to $[0,u-1]$ is one-to-one. This is a $\Pi_1$ formula derivable in $\text{IA}_0^{+\text{Exp}}$ but very likely not derivable in $\text{IA}_0$ itself (maybe even not in $\text{IA}_0^{+\Omega_1}$), see [A]. A weaker version of PHP which says only that there is no one-to-one mapping from $[0,2u]$ to $[0,u]$, i.e.

$\forall u \left( \forall x \leq 2u \exists y < u \varphi(x,y) \rightarrow \exists x_0 < x \leq 2u \exists y < u \varphi(x_0,y) \& \varphi(x_1,y) \right)$,

is derivable in $\text{IA}_0^{+\Omega_1}$, see [FWW]. We shall denote it by $\Delta_0$-$\text{WPHP}$.

The difficulty of proving $\Delta_0$-$\text{PHP}$ is caused by the fact that there is no $\Delta_0$ definition for counting the number of elements that satisfy a $\Delta_0$ formula. (This follows from a result of Toda [T], provided that Polynomial Hierarchy does not collapse). Thus there is another interesting question: what are all arithmetical consequences of "having counting functions". This could be formalized by adding a new function symbol for each $\Delta_0$ formula with a corresponding axiom and by extending the schema of induction to bounded formulas containing also the new symbols. PHP is a typical statement derivable using counting.

A natural generalization of the Pigeon Hole Principle is the finite Ramsey theorem. This is usually stated as follows. Let $k, m, r$ be positive numbers, $k < m$, then there exists $n$ such that if we color the $k$ element subsets of an $n$ element set $X$ by $r$ colors, then there exists $Y$, an $m$ element
number \( n \) must be exponential in \( m \), thus it cannot be presented as a \( \Pi_1 \) sentence.

We shall consider a \( \Pi_1 \) form of the finite Ramsey's theorem. In order to simplify the matter let us take the special case of \( k=2 \) and \( r=2 \). Then for each coloring of pairs on an \( n \) element set \( X \) by two colors there is a set \( Y \) of size \( \lceil \frac{1}{2} \log_2 n \rceil \) such that all pairs of elements of \( Y \) have the same color. (The bound to the size \( \lceil \frac{1}{2} \log_2 n \rceil \) is not the best one, we shall discuss this question later). We formalize this statement as follows: For every bounded formula \( \varphi(x,y,z) \), possibly with other parameters,

\[
\forall u \exists z < 2 \varphi(x,y,z) \rightarrow \exists z < 2 \exists Y \subseteq u \left( |Y| = \lceil \frac{1}{2} \log_2 u \rceil \land \forall x, y \in Y (x < y \rightarrow \varphi(x,y,z)) \right).
\]

This is not a \( \Pi_1 \) formula, however if we extend the language by a function symbol for \( x \log_2 x \), we can talk about subsets of \( u \) of size \( \log_2 u \). This is quite justified in \( I\Delta_0 + \Omega_1 \), since the axiom \( \Omega_1 \) just says that \( x \log_2 x \) is a total function. We can work also in the equivalent theory \( S_2 \) which has the smash function \( x\# y \) with the same growth rate as \( x \log_2 x \), see [B1]. For larger \( k \) and \( r \) the set \( Y \) is even smaller, thus also the general finite Ramsey theorem can be formalized in this way.

The purpose of this paper is to show that such formalizations of finite Ramsey's theorem are derivable from the Pigeon Hole Principle. In fact they are derivable from the weaker version \( \Delta_0 \)-WPHP, hence they are provable in \( I\Delta_0 + \Omega_1 \). Using a translation of bounded formulas into propositional formulas we conclude that propositional formulas expressing the finite Ramsey theorem have polynomial-size bounded-depth Frege proofs. For a discussion of related questions on propositional proof systems see section 4. We shall also discuss some complexity theoretical questions concerning Ramsey's theorem and present principles, which are candidates for natural counting principles not derivable from the Pigeon Hole Principle.

2 The proof of the Ramsey schema

We shall prove the Ramsey schema for pairs and then, in the next section, sketch the proof in the general case. We shall use the language of arithmetic extended by \( x \log_2 x \), thus in bounded (i.e. \( \Delta_0 \)) formulas we can quantify sets of logarithmic size.

Theorem 1

The following sentence is provable in \( I\Delta_0 + \Omega_1 \) for every \( \Delta_0 \) formula \( \varphi(x,y,z) \):

\[
\forall u \forall r \exists z < r \exists Y \subseteq u \left( |Y| = \lceil \frac{1}{2} \log_2 u \rceil \land \forall x, y \in Y (x < y \rightarrow \varphi(x,y,z)) \right).
\]