Some Constructions in Rings of Differential Polynomials*

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Introduction

In the recent years, motivated by the success of various algorithms and techniques in computational algebraic ring theory, the researchers in the Computer Algebra community have turned their attention to the task of extending these ideas to differential ring. Appropriate analogs and generalizations of standard bases, Buchberger's completion algorithm, etc. are however yet to be found. (See for example [1], [2], [16] and [19]).

The reasons for this interest are both practical and theoretical. There is, in fact, a growing effort to use differential algebra in order to solve problems in Control theory, Dynamical systems and Robotics. From the theoretical point of view, it is equally important that we understand the precise relation between 'old' constructive methods (Ritt-Seidenberg algorithm [5,24]) and the recent Gröbner bases-like approach.

A constructive study of differential algebra may also give new insight into its quite complicated structures. For example, rings of differential polynomials are not Noetherian, hence differential ideals can be much more complex than algebraic ideals. An example, reported later on, shows that this difference implies that there are differential ideals that are not recursive!

On the other hand the structure of differential ideals is not completely unruly, and one can hope to characterize classes of differential rings and of ideals for which suitable algorithmic techniques can be developed. The concept of H-bases for differential ideals is introduced in the second part of this paper and it constitutes a contribution to this direction of research.

The differential algebras considered here are commutative rings of differential polynomials in several differential indeterminates over a field of constants, a particular case of the algebras introduced by the classical works of Janet [8], Kolchin [11], Riquier [20].

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and Ritt [22]. These objects should not be confused with the rings of the differential operators whose constructive aspects have been studied, for example, in [3] and [4].

The rest of this paper is organized as follows: The first section fixes the standard notations and recalls some classical notions in Differential Algebra. The second section presents an example of a nonrecursive differential ideal in the ring \( \mathbb{Z}\{x\} \). This example, which generalizes the one due to Ritt and Kolchin, shows that the problem of deciding the membership in a differential ideal is not in general algorithmically solvable. It however leaves open the question whether the membership question for a recursively generated differential ideal can be solved by a recursive procedure. The third section introduces the concept of H-bases for differential ideal, in analogy with Macaulay's original definition of H-bases for algebraic ideals [13] (see also [15] and [23]).

These bases can be regarded as a special kind of standard bases (cfr. [1], [16] and [17]). Using such bases, one can introduce a particularly simple procedure to test the membership of a differential polynomial in a differential ideal. Unfortunately, for general differential ideals, it is not known if there is a finite method capable of computing an H-basis, or even a useful subset of it.

In the special case of ideals with an isobaric basis, however, H-bases can be computed and we present an effective procedure to decide the membership in such ideals. It is still unresolved if a finitely generated (respectively, recursively generated) differential ideal has a finite (respectively, recursive) H-basis.

1 Preliminaries

In order to keep the following expositions reasonably self-contained, we shall introduce in this section most of the standard notations to be used later. For those notations and definitions that are not explicitly mentioned here, refer to [9], [11] or [22].

All our rings are assumed to be commutative and with unity.

Definition 1.1 A ring \( R \) is said to be a differential ring if there exists a differential operator from \( R \) to \( R \), i.e., a map \( d: R \to R \) such that, for all \( \alpha \) and \( \beta \) in \( R \):

- \( d \) is linear, i.e. \( d(\alpha + \beta) = d(\alpha) + d(\beta) \);
- \( d(\alpha \beta) = d(\alpha)\beta + \alpha d(\beta) \).

Example 1.1 Any algebraic ring \( R \), with \( d \) defined as the trivial derivation, i.e. \( d(r) = 0 \) for all \( r \in R \), is a differential ring.

Example 1.2 The ring of analytic functions over a domain of \( \mathbb{C} \) is a differential ring.

Definition 1.2 A subset \( I \) of a differential ring \( R \) is a differential ideal if it is an algebraic ideal of \( R \) and moreover, it is closed under the \( d \) operator, i.e. if \( d(I) \subseteq I \).

If \( S \) is a subset of \( R \) and \( I \) is the minimal differential ideal of \( R \) containing \( S \), then \( S \) is said to be a system of generators for \( I \), or equivalently \( I \) is said to be the ideal generated by \( S \). If \( S = \{f_1, \ldots, f_n\} \) then \( I \) is denoted by \([f_1, \ldots, f_n]\).

Since \( R \) is a particular algebraic ring one can also consider the algebraic ideal \( J \) generated by \( S \). This ideal is sometimes denoted by \((f_1, \ldots, f_n)\). Note that, in general, \((f_1, \ldots, f_n) \neq [f_1, \ldots, f_n] \).