Uncertainty in the valuation of risky assets

Risky financial assets are defined by their future random payments or their returns, in most modern Finance models. Under the hypothesis that markets do not allow arbitrage opportunities, it has been shown in modern Finance models (Black and Scholes' [1973] famous option valuation formula for instance) that assets are valuated by the expectation of their random payments with respect to a probability distribution. This distribution is called the implicit distribution because it is not the objective distribution that was used to describe uncertainty. Because there are no arbitrages, the value of an asset is a linear function of its random payments, and under some assumptions a Riesz decomposition theorem can be used to express this linear functional as a mathematical expectation.

In the first part of the paper we indicate which No Arbitrage assumptions (we distinguish two) give the linearity and the positivity necessary to use the decomposition theorem when considering a probability space $(S, \mathcal{A}, P)$ where $S$, the set of states, is not necessarily finite. The precise result we obtain is important because it allows to give an economic foundation to general Finance models under the assumption that uncertainty is described by a probability space. Such an economic foundation had already been given (Arrow [1953]) in a more general setting, since in Arrow's model uncertainty is described merely through a measurable space $(S, \mathcal{A})$ (no probability distribution is given; on the other hand there is some restriction since only a finite $S$ was considered).

We keep this type of generality in the second part of the paper where we define an asset by the list of its future payments. Future payments are contingent to states of nature in a given measurable space $(S, \mathcal{A})$ but we do not assume it is probabilized, nor finite. First we show that translating No Arbitrage assumptions in this non probabilized setting, leads again to a valuation of assets in terms of mathematical expectation. Then we consider an asset valuation founded on the representation of an ordering consistent with market prices. Market prices rank marketed assets; this ranking is extended to the set of all assets under some assumptions related to the way the market works. One assumption can give a mathematical expectation valuation of future payments in the same way the No Arbitrage assumptions did in equilibrium models. However it can be weakened so as to allow some kind of arbitrage opportunities which can be observed in real markets. On the mathematical level this last model is interesting by its representation of an ordering which can be less stringent than the mathematical expectation obtained under no arbitrage. Because arbitrage possibilities leave some indeterminacy in the formation costs of portfolios replicating assets, the valuation of these assets is not expressed by a mathematical expectation but by a Choquet's integral with respect to a capacity (we use Schmeidler's [1986], [1989] and Yaari's [1987] models). This capacity is implicitly revealed by market prices, prices which, because of arbitrage possibilities, leave some indeterminacy upon assets valuation. The implicit capacity, like the implicit probability in the No Arbitrage theory, is the market appreciation of uncertainty on random payments. When a market is complete, i.e. when there is a sufficient number of marketed assets to replicate any relevant payment scheme by way of a linear combination of marketed assets (portfolio), the implicit distribution is unique and can be calculated using market prices. The determination of a capacity requires many more data than the determination of a probability, this is the price to pay for the indeterminacy arbitrage possibilities leave upon assets valuation.

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1. The case of probabilized uncertainty.

1.1. Arrow's model of assets market with a finite set of states of the world.

Arrow [1953] built up a model in which agents, using assets portfolios, could achieve their optimal consumptions in any uncertain states of the world. Arrow's assets pay one unit of money if a certain state of the world obtains, nothing otherwise. Prices of these assets (one asset for each state of the world) are positive and can be normalized so that they sum to one. Hence they can be interpreted as a probability distribution on the set of states. This distribution is not an objective distribution, even if such an objective distribution exists, nor a subjective distribution agents could assess to the states in order to make their decisions. It is a weighting of the states made by prices which express an aggregation of agents behavior towards uncertainty.

A side result of this model is that assets values are their expected payoffs with respect to the distribution revealed by prices. This is obtained under a "No Arbitrage" assumption (implicit in all equilibrium models): "two assets that are the same can't sell at different prices". This implies that the value of a portfolio is its formation cost, which, in turn implies that the value functional (defined on the set of assets: real valued functions on the set of states) is linear. The functional is positive because of an other "No Arbitrage" assumption: an asset with positive payments must have a non negative value.

1.2. Assets valuation in modern finance models.

Arrow's model and its conclusions can be generalized so as to encompass valuation models in finance, notably the famous one by Black and Scholes. These models assume uncertainty to be described by a probability space \((S, \mathcal{A}, \mu)\) where \(\mu\) is known. For instance in the Black and Scholes model, \(\mu\) is such that the process of future random prices (of the underlying security) is a generalized Wiener process with a constant drift and known instantaneous variance.

The set \(Y\) of assets can be taken to be the space \(L^2(S, \mathcal{A}, \mu)\) endowed with the \(L^2\) norm topology. Let \(M (Y \supseteq M)\) denotes the set of marketed assets, and assume the characteristic function \(1_S\) of \(S\), belongs to \(M\) (the riskless asset paying one unit of money in all states). Let \(\Theta\) denote the set of portfolios \(\theta\) which can be built with marketed assets. By definition a portfolio \(\theta\) is defined as the list \(\theta(y) \in \mathbb{R}, y \in M\) of quantities of marketed asset \(y\) with which portfolio \(\theta\) is formed, where only a finite number of \(\theta(y)\) is different from zero. Such a portfolio \(\theta\) uniquely yields an asset, say \(\theta^*\) the payments of whom satisfy \(\theta^* = \sum_{y \in M} \theta(y) y\). Notice that the set \(\{\theta^*, \theta \in \Theta\} = \text{span}(M)\). \(\text{span}(M)\) will be referred to as the set of marketable assets and we assume that the market is complete, more precisely we assume that \(\text{span}(M) = Y\). Marketed assets \(y \in M\) have prices \(q(y)\), and the formation cost of a portfolio defined by \((\theta(y))_{y \in M}\), is: \(K(\theta) = \sum_{y \in M} \theta(y) q(y)\). The No Arbitrage assumptions bear upon portfolios.

No Arbitrage1:
Let \(\theta\) and \(\theta'\) be two portfolios such that \(\theta^* = \theta'^*\), then \(K(\theta) = K(\theta')\).